THE

ASTROPHYSICAL JOURNAL

An International Review of Spectroscopy and Astronomical Physics

FOUNDED IN 1895 BY GEORGE E. HALE AND JAMES E. KEELER

VOLUME 90

JULY–DECEMBER, 1939

THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS

THE CAMBRIDGE UNIVERSITY PRESS, LONDON
THE MARUZEN COMPANY LIMITED, TOKYO
THE COMMERCIAL PRESS LIMITED, SHANGHAI

PUBLISHED JULY, SEPTEMBER, OCTOBER, NOVEMBER, DECEMBER 1939

COMPOSED AND PRINTED BY THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS, U.S.A.

CONTENTS

NUMBER 1

NUMBER 2

NUMBER 3

NUMBER 4

NUMBER 5

THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

VOLUME 90 — JULY 1939 — NUMBER 1

THE DYNAMICS OF STELLAR SYSTEMS. I–VIII*

S. CHANDRASEKHAR

ABSTRACT

The kinematical characteristics which are postulated in this paper for describing the state of motions in a stellar system are the following: (i) For every small region in the system we can define uniquely a *local standard of rest.* (ii) The different local standards of rest are in *relative motion.* (iii) The motions of the individual stars with respect to the appropriate local frames of references are distributed according to a *generalized ellipsoidal law.* (iv) The motions of the individual stars are governed by a potential function $\mathfrak{V}$. And, finally (v), the system is in a steady state. The present investigation is concerned with the dynamics of such stellar systems with *differential motions.*

Mathematically, the dynamical problem reduces to finding the circumstances under which the equation of continuity regarded as a partial differential equation for the *distribution function*, Ψ, admits of a solution of the form

$$\Psi \equiv \Psi(Q + \sigma[x, y, z]) \ , \tag{i}$$

where Ψ is an arbitrary function of the argument and Q is defined by

$$\left.\begin{aligned} Q = a(U - U_0)^2 + b(V - V_0)^2 + c(W - W_0)^2 + 2f(V - V_0)\,(W - W_0) \\ + 2g\,(W - W_0)\,(U - U_0) + 2h\,(U - U_0)\,(V - V_0) \ , \end{aligned}\right\} \tag{ii}$$

and σ is related to the star density at the point considered. In equation (ii), a, b, c, f, g, and h are the coefficients of the velocity ellipsoid, (U_0, V_0, W_0) and (U, V, W) define the motions of the local centroids and the individual stars with respect to a fundamental frame of reference, respectively. In the mathematical analysis it is assumed that $a, b, c, f, g, h, U_0, V_0, W_0$, and σ can be arbitrary continuous functions of position.

The dynamical problem is formulated in Part I. It is shown here that the mathematical problem reduces to the discussion of four groups of equations: (i) a set of ten simultaneous partial differential equations which involve the coefficients of the velocity ellipsoid only; (ii) a set of six equations which will determine the motion of the local centroid as a function of position; (iii) a linear homogeneous partial differential equation of the Lagrangian type for the potential function $\mathfrak{V}$, and (iv) a set of three equations which lead to three other integrability conditions.

In Part II the complete solution for the two-dimensional problem is given. In this

* This paper was presented at the symposium on "Galactic and Extragalactic Structure" in connection with the dedication of the McDonald Observatory, May 5–8, 1939.

part most of the known results of stellar dynamics are derived as special cases of the general theory. In Part III the first two groups of sixteen equations are considered in Cartesian co-ordinates. It is found that the solution for a, b, c, f, g, and h involves twenty constants of integration; the solutions for the motion similarly involve six further constants of integration. In Part IV a fundamental theorem in the dynamics of stellar systems is proved. It is shown here that stellar systems with differential motions must necessarily be characterized by a *helical symmetry* of the potential function $\mathfrak{V}$.

In Part V the problem is discussed in spherical polar co-ordinates, and the complete enumeration of the different types of stellar systems which are characterized by a spherically symmetrical potential function $\mathfrak{V}$ is made. In Part VI the problem is considered in spheroidal co-ordinates, and certain special types of stellar systems are treated.

In Part VII we discuss the problem in cylindrical co-ordinates. Special classes of stellar systems with axial symmetries are considered in this part. The motions inside a uniform spheroidal distribution of mass is treated; this discussion discloses the existence of two critical spheroids in which the motions of the local centroids have components perpendicular to the galactic plane. One of these two critical spheroids is an oblate spheroid with a ratio of the axes of about 3.41 : 1; it is suggested that the existence of this critical value for the ratio of the axes is probably connected with the known upper limit to the eccentricities of elliptical nebulae. Finally, in Part VIII the general theory of ellipsoidal systems is described.

INTRODUCTION

1. As is well known, our knowledge of the state of stellar motions in the galaxy is summarized in the phenomenon of star streaming, on the one hand, and in the Oort-Lindblad theory of differential galactic rotation, on the other. These two distinct aspects of the kinematics of the stellar motions which are the content of the theory of star streaming and the theory of galactic rotation, respectively, are really consequences of the two possible standards of rest with respect to which we can describe the motions in a dynamical system like our galaxy. More explicitly, the two standards of rest that are contemplated are the following:

Consider a dynamical system of a large number of particles.[1] Then there is a unique standard of rest defined by the motions of all the particles in the assembly. This is the standard of rest contemplated, for instance, when we solve for the solar motion with respect to the system of the globular clusters.[2] We shall refer to the standard of rest, defined by all the motions in the system, as the "fundamental standard of rest," and to the corresponding frame of reference as the "fundamental frame of reference."

On the other hand, if we restrict ourselves to a small region of

[1] Stars in the present connection.

[2] See the remarks in the author's paper in *M.N.*, **98**, 710, 1938—esp. p. 712 n.

space in the neighborhood of a given point, then the particles in this region of space will enable us to define a corresponding standard of rest. The standard of rest contemplated when we solve for the solar motion with respect to the "near-by stars" is of this nature. In order, however, to avoid the ambiguity in the notion of the "near-by stars," we shall assume that, as we make the volume of the region of space considered tend to zero, we shall obtain (in the limit) a unique standard of rest. We shall refer to the standard of rest defined in this manner as the "local standard of rest" at the point considered. It is clear that the local standard of rest and the corresponding "local frame of reference" will be functions of position. In particular, the different local standards of rest at different points will, in general, be in relative motion.

To fix our ideas, let us imagine the fundamental frame of reference to be specified by a Cartesian system of co-ordinates x, y, and z. Let U, V, and W denote the components of the velocity of a star with respect to the fundamental frame of reference. Consider the motions in a small region of space Σ surrounding the point (x, y, z). The stars in the volume element Σ will define a certain standard of rest. We assume that, as we make Σ tend to zero, we shall obtain a standard of rest which is independent of the manner[3] in which we may let Σ tend to zero. The standard of rest thus uniquely defined at the point (x, y, z) is the local standard of rest at (x, y, z). Let U_0, V_0, and W_0 denote the components of the velocity of the local standard at (x, y, z) with respect to the fundamental standard of rest. We shall refer to U_0, V_0, and W_0 as the components of the "motion of the local centroid" at (x, y, z). It is clear from the manner in which we have derived them that U_0, V_0, and W_0 will, in general, be functions of position. As has been shown by Ogrodnikoff and Milne,[4] it is the dependence and the variation of U_0, V_0, and W_0 with position that give rise to the phenomenon of differential motions in the system.

If U, V, and W are the components of the velocity of an individual star at (x, y, z) and (U_0, V_0, W_0) defines the motion of the local cen-

[3] I.e., the sequence of shapes.

[4] For references see W. M. Smart, *Stellar Dynamics*, pp. 405–410, Cambridge, Eng. 1938.

troid at (x, y, z), then the components u, v, and w of the residual motion of the star are given by

$$u = U - U_0\,; \qquad v = V - V_0\,; \qquad w = W - W_0\,. \tag{1}$$

Now, returning to the kinematics of stellar motions, we see that in our present terminology, the phenomenon of star streaming is one which we encounter when we analyze the residual motions. In the same way, the theory of differential galactic rotation arises when we analyze the motions of local centroids. More explicitly, the characteristic features of the two descriptions can be summarized as follows:

The analysis of the radial velocities and of the proper motions of the near-by stars has shown that the residual motions (u, v, w) are characterized by (*a*) *randomness*, i.e., in any given direction the number of positive velocities (in any given range) equals the number of negative velocities (in an equal range), and (*b*) *a direction of preferential motion*, i.e., the mean speed in this direction is the maximum. (It is this direction of maximum mean residual motion that defines the vertex of star streaming.) Quantitatively, these ideas are included in Schwarzschild's hypothesis of the ellipsoidal law of the distribution of velocities, according to which the number of stars $d\mathfrak{N}$ with residual motions in the velocity range $(u, u + du; v, v + dv; w, w + dw)$ and in the element of volume $dxdydz$ is given by

$$d\mathfrak{N} = \rho(x, y, z)e^{-Q}dudvdwdxdydz\,, \tag{2}$$

where ρ is related to the star density at the point considered and

$$Q = au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv\,. \tag{3}$$

In equation (3) a, b, c, f, g, and h are the coefficients of the velocity ellipsoid and will be functions of position. The directions of the principal axes of the velocity ellipsoid can be obtained by the rotation which would be necessary to bring the quadratic form (3) to its normal form.

As is well known, a prolate spheroidal distribution of velocities is entirely sufficient to adequately represent the observational ma-

terial. Further, if we restrict ourselves to stars near the galactic plane (which we shall assume to coincide with the XY-plane of the fundamental frame of reference[5]) then the principal axes of the velocity spheroid are in the radial (Π), the transverse (Θ), and the perpendicular (Z) directions, the longest axis being in the radial direction. As we shall see (cf. § 9), this implies that Q has the simplified form

$$Q = (\kappa_2 y^2 + \kappa_1)u^2 + (\kappa_2 x^2 + \kappa_1)v^2 - 2\kappa_2 xy\, uv + \kappa_3 w^2 \,, \tag{4}$$

In the discussions relating to the phenomenon of star streaming, (u, v, w) refers, of course, to the residual motion $(U - U_0, V - V_0, W - W_0)$. To complete our kinematics we need to describe the character of the differential motions. According to the Oort-Lindblad theory, the motions of the local centroids (at any rate, near the galactic plane) are predominantly of a rotational kind, i.e., we can write

$$\left.\begin{aligned} U_0 &= -\Theta_0 \sin\theta \,, \\ V_0 &= \Theta_0 \cos\theta \,, \\ W_0 &= 0 \,, \end{aligned}\right\} \tag{5}$$

where Θ_0 is, in the first instance, an arbitrary function of the distance from the galactic center.[6]

From this summary of the observed kinematics of the stellar motions in the galaxy, it is clear that the state of motions encountered in the neighborhood of the sun is a very special case of a more general character in which the components of the motions of the local centroid, U_0, V_0, and W_0, are allowed to be arbitrary continuous functions of position. Further, it is unnecessary to restrict needlessly the distribution function to the form (2). Instead we shall assume for it the more general form

$$d\mathfrak{N} = \Psi(Q + \sigma)dxdydzdudvdw \,, \tag{6}$$

[5] The origin of the system of co-ordinates is further assumed to be at the galactic center.

[6] There are some restrictions on the nature of the function Θ_0 from the dynamics of the situation (cf. § 9).

where Ψ is allowed to be an arbitrary function of the argument $Q + \sigma$ and σ is an arbitrary function of the space co-ordinates; for a Gaussian form for the distribution function, σ will be related to ρ according to (cf. Eq. [2])

$$\rho = e^{-\sigma} \tag{6'}$$

We shall assume further that the six coefficients of the velocity ellipsoid (a, b, c, f, g, h) are continuous functions of position. We shall refer to systems having the features described here as "systems with differential motions." It is the object of this present investigation to study in detail the dynamics of stellar systems with differential motions.

The plan of this paper is as follows:

In Part I the precise mathematical formulation of the dynamical problem is given, and a fundamental set of twenty simultaneous partial differential equations is derived. It is the discussion of this system of differential equations which leads to the theory of stellar systems with differential motions.

In Part II the complete solution for the two-dimensional problem is given. Many known results are here obtained under more general circumstances and as special cases.

In Part III the general solutions for the coefficients of the velocity ellipsoid and the motions of the local centroids are obtained in Cartesian co-ordinates. The general solutions are shown to involve twenty and six arbitrary constants of integration, respectively.

In Part IV a fundamental theorem in the dynamics of stellar systems with differential motions is proved. It is shown in this part that such stellar systems must necessarily be characterized by an axis of helical symmetry.

In Part V the fundamental differential equations are discussed in spherical polar co-ordinates, and the complete enumeration of stellar systems with spherical symmetry for the potential is made. The theory presented here should prove adequate as a basis for the dynamics of globular clusters.

In Part VI the discussion is carried out in spheroidal co-ordinates. This part is, perhaps, the most "difficult" in the present investigation.

In Part VII the solutions of the fundamental equations in cylindrical co-ordinates are obtained. Stellar systems with certain types of axial symmetries are considered.

In Part VIII the general theory of stellar systems for which the equipotential surfaces are concentric ellipsoids (or spheroids) is considered.

In Parts IX and X (now under preparation) the theory of stellar systems with helical symmetry and the theory of nonsteady states are considered. In Part XI (also under preparation) the physical interpretation of the results of the present investigation will be considered at some length.

I. THE GENERAL THEORY

2. *The general formulation of the dynamical problem.*—In the introductory section we developed the general notion of differential motions by choosing for the fundamental frame of reference a Cartesian system of co-ordinates. However, in several of the problems that we shall encounter it will be more convenient to choose other systems of co-ordinates, e.g., spherical polar, spheroidal, cylindrical, etc. We shall therefore formulate the dynamical problem in a general orthogonal curvilinear system of co-ordinates (λ, μ, ν).[7]

Now a general orthogonal system of co-ordinates defines a triple infinity of one-parametric families of surfaces such that through any given point there pass three surfaces belonging, one each, to the three families. By definition the three curves of intersection (of the three surfaces passing through any given point) are mutually orthogonal and define the principal directions at the point considered. Further, the values of the parameters characterizing the three surfaces passing through a given point are the curvilinear co-ordinates λ, μ, and ν of the point. If we consider the six surfaces (λ, μ, ν) and $(\lambda + d\lambda, \mu + d\mu, \nu + d\nu)$, they will bound an infinitesimal parallelepiped the lengths of whose edges can be written as

$$P d\lambda \; ; \qquad Q d\mu \; ; \qquad R d\nu \, , \tag{7}$$

[7] It should be emphasized that the choice of the system of co-ordinates is merely a matter of convenience. The physical problem cannot be altered by the choice of a system of co-ordinates.

where P, Q, and R will, in general, be functions of λ, μ, and ν. If x, y, and z are known as functions of λ, μ, and ν, the expressions for determining P, Q, and R are given by

$$\left.\begin{aligned} P^2 &= \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2 , \\ Q^2 &= \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2 , \\ R^2 &= \left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2 + \left(\frac{\partial z}{\partial \nu}\right)^2 . \end{aligned}\right\} \qquad (8)^{8}$$

Let the components of the velocity of a star at (λ, μ, ν) along the principal directions at (λ, μ, ν) be denoted by Λ, M, and N. By (7) we have

$$\Lambda = P\dot{\lambda} \; ; \qquad \mathrm{M} = Q\dot{\mu} \; ; \qquad \mathrm{N} = R\dot{\nu} \; . \qquad (9)$$

We now characterize a stellar system with differential motions as follows:

I. *At any given point* (λ, μ, ν) *we can define uniquely a local standard of rest which is a continuous function of position and time.*

Let Λ_0, M_0, and N_0 denote the components of the motion of the local centroid at (λ, μ, ν) along the principal directions at (λ, μ, ν). According to our assumption, Λ_0, M_0, and N_0 are continuous functions of λ, μ, and ν and of the time t.

The components of the residual motion of a star at (λ, μ, ν) along the principal directions are clearly

$$\Lambda - \Lambda_0 \; ; \qquad \mathrm{M} - \mathrm{M}_0 \; ; \qquad \mathrm{N} - \mathrm{N}_0 \; . \qquad (10)$$

Let us denote the distribution function by $\Psi(\lambda, \mu, \nu; \Lambda, \mathrm{M}, \mathrm{N}; t)$ so that the number of stars $d\mathfrak{N}$ at (λ, μ, ν) with space co-ordinates in the range $(\lambda, \lambda + d\lambda; \mu, \mu + d\mu; \nu, \nu + d\nu)$ and the velocity co-

[8] For an orthogonal system of co-ordinates the "cross-coefficients"

$$\sum_{x,y,z} \frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \mu} , \qquad \sum_{x,y,z} \frac{\partial x}{\partial \mu}\frac{\partial x}{\partial \nu} , \qquad \sum_{x,y,z} \frac{\partial x}{\partial \nu}\frac{\partial x}{\partial \mu}$$

should all be zero.

ordinates in the range (Λ, $\Lambda + d\Lambda$; M, M + dM; N, N + dN) at time t is given by

$$d\mathfrak{N} = \Psi(\lambda, \mu, \nu; \Lambda, M, N; t) PQR d\lambda d\mu d\nu\, d\Lambda dM dN \,. \tag{11}$$

Our second assumption is:

II. *The distribution function* $\Psi(\lambda, \mu, \nu; \Lambda, M, N; \mathrm{t})$ *is of the generalized Schwarzschild type, i.e.,*

$$\Psi(\lambda, \mu, \nu; \Lambda, M, N; t) \equiv \Psi(Q + \sigma) \,, \tag{12}$$

where Q *stands for*

$$\left.\begin{aligned} Q = a(\Lambda - \Lambda_0)^2 + b(M - M_0)^2 + c(N - N_0)^2 \\ + 2f(M - M_0)(N - N_0) + 2g(N - N_0)(\Lambda - \Lambda_0) \\ + 2h(\Lambda - \Lambda_0)(M - M_0) \,; \end{aligned}\right\} \tag{13}$$

further, the coefficients of the velocity ellipsoid, a, b, c, f, g, *and* h *and the function* σ *are all continuous functions of position* λ, μ, *and* ν *and of the time*, t.

Our third assumption is:

III. *The motions of the individual stars are governed by a potential function* $\mathfrak{B}(\lambda, \mu, \nu; \mathrm{t})$ *per unit mass.*

This third assumption implies that the distribution function Ψ satisfies the equation of continuity

$$\frac{\partial\Psi}{\partial t} + \dot{\lambda}\frac{\partial\Psi}{\partial\lambda} + \dot{\mu}\frac{\partial\Psi}{\partial\mu} + \dot{\nu}\frac{\partial\Psi}{\partial\nu} + \dot{\Lambda}\frac{\partial\Psi}{\partial\Lambda} + \dot{M}\frac{\partial\Psi}{\partial M} + \dot{N}\frac{\partial\Psi}{\partial N} = 0 \,, \tag{14}$$

or by (9)

$$\frac{\partial\Psi}{\partial t} + \frac{\Lambda}{P}\frac{\partial\Psi}{\partial\lambda} + \frac{M}{Q}\frac{\partial\Psi}{\partial\mu} + \frac{N}{R}\frac{\partial\Psi}{\partial\nu} + \dot{\Lambda}\frac{\partial\Psi}{\partial\Lambda} + \dot{M}\frac{\partial\Psi}{\partial M} + \dot{N}\frac{\partial\Psi}{\partial N} = 0 \,, \tag{15}$$

where the "accelerations" $\dot{\Lambda}$, $\dot{M}$, and $\dot{N}$ are to be expressed in terms of Λ, M, N, λ, μ, and ν by using the equations of motion (see § 3, below).

The dynamical problem is: *Under what circumstances will the equation of continuity (15), regarded as a partial differential equation for* Ψ, *admit of a solution of the form (12)?*

In the present paper we shall restrict ourselves to a consideration of stellar systems with differential motions which are in a steady state; i.e., we shall make the additional assumption:

IV. *The motion of the local centroid,* $(\Lambda_0, \mathrm{M}_0, \mathrm{N}_0)$, *the distribution function,* Ψ, *the "density function,"* σ, *and the coefficients of the velocity ellipsoid,* (a, b, c, f, g, h), *are all functions of the position,* (λ, μ, ν), *only and are independent of the time,* t.

For systems in a steady state the partial differential equation for Ψ reduces to

$$\frac{\Lambda}{P}\frac{\partial\Psi}{\partial\lambda}+\frac{\mathrm{M}}{Q}\frac{\partial\Psi}{\partial\mu}+\frac{\mathrm{N}}{R}\frac{\partial\Psi}{\partial\nu}+\dot{\Lambda}\frac{\partial\Psi}{\partial\Lambda}+\dot{\mathrm{M}}\frac{\partial\Psi}{\partial\mathrm{M}}+\dot{\mathrm{N}}\frac{\partial\Psi}{\partial\mathrm{N}}=0\,. \tag{16}$$

We now proceed to write down explicitly the conditions that (16) may admit of a solution of the form

$$\Psi \equiv \Psi(Q+\sigma)\,, \tag{17}$$

where Q stands for the quadratic form (13), with Λ_0, M_0, N_0, a, b, c, f, g, and h now being independent of the time, t.

3. *The equations of the problem.*—The first thing we shall have to do is to express the accelerations $\dot{\Lambda}$, $\dot{\mathrm{M}}$, and $\dot{\mathrm{N}}$ in terms of the space (λ, μ, ν) and velocity $(\Lambda, \mathrm{M}, \mathrm{N})$ co-ordinates by using the equations of motion in the Lagrangian form. If $\mathfrak{T}$ and $\mathfrak{V}$ are the kinetic and potential energies (per unit mass) of the star, then the Lagrangian equations are

$$\frac{d}{dt}\left(\frac{\partial\mathfrak{T}}{\partial\dot{\lambda}}\right)-\frac{\partial\mathfrak{T}}{\partial\lambda}=-\frac{\partial\mathfrak{V}}{\partial\lambda} \tag{18}$$

and two similar equations. Now, according to (9),

$$\mathfrak{T}=\tfrac{1}{2}(P^2\dot{\lambda}^2+Q^2\dot{\mu}^2+R^2\dot{\nu}^2)\,, \tag{19}$$

so that the momenta co-ordinates conjugate to (λ, μ, ν) are

$$p_\lambda = P^2\dot{\lambda}\,; \qquad p_\mu = Q^2\dot{\mu}\,; \qquad p_\nu = R^2\dot{\nu}\,. \tag{20}$$

Equation (18) now reduces to

$$\frac{d}{dt}(P^2\dot{\lambda}) - \frac{1}{2}\left(\dot{\lambda}^2\frac{\partial P^2}{\partial\lambda} + \dot{\mu}^2\frac{\partial Q^2}{\partial\lambda} + \dot{\nu}^2\frac{\partial R^2}{\partial\lambda}\right) = -\frac{\partial\mathfrak{V}}{\partial\lambda}. \tag{21}$$

But

$$\frac{d}{dt}(P^2\dot{\lambda}) = P^2\ddot{\lambda} + \dot{\lambda}\left(\dot{\lambda}\frac{\partial P^2}{\partial\lambda} + \dot{\mu}\frac{\partial P^2}{\partial\mu} + \dot{\nu}\frac{\partial P^2}{\partial\nu}\right). \tag{22}$$

Combining (21) and (22), we have

$$P^2\ddot{\lambda} = -\tfrac{1}{2}\dot{\lambda}^2\frac{\partial P^2}{\partial\lambda} + \tfrac{1}{2}\dot{\mu}^2\frac{\partial Q^2}{\partial\lambda} + \tfrac{1}{2}\dot{\nu}^2\frac{\partial R^2}{\partial\lambda} - \dot{\lambda}\dot{\mu}\frac{\partial P^2}{\partial\mu} - \dot{\lambda}\dot{\nu}\frac{\partial P^2}{\partial\nu} - \frac{\partial\mathfrak{V}}{\partial\lambda}. \tag{23}$$

On the other hand,

$$P\dot{\Lambda} = P\frac{d}{dt}(P\dot{\lambda}) = P^2\ddot{\lambda} + \tfrac{1}{2}\dot{\lambda}\frac{dP^2}{dt}, \tag{24}$$

or

$$P\dot{\Lambda} = P^2\ddot{\lambda} + \tfrac{1}{2}\dot{\lambda}\left(\dot{\lambda}\frac{\partial P^2}{\partial\lambda} + \dot{\mu}\frac{\partial P^2}{\partial\mu} + \dot{\nu}\frac{\partial P^2}{\partial\nu}\right). \tag{25}$$

Substituting for $P^2\ddot{\lambda}$ from (23), we obtain

$$P\dot{\Lambda} = \tfrac{1}{2}\dot{\mu}^2\frac{\partial Q^2}{\partial\lambda} + \tfrac{1}{2}\dot{\nu}^2\frac{\partial R^2}{\partial\lambda} - \tfrac{1}{2}\dot{\lambda}\dot{\mu}\frac{\partial P^2}{\partial\mu} - \tfrac{1}{2}\dot{\lambda}\dot{\nu}\frac{\partial P^2}{\partial\nu} - \frac{\partial\mathfrak{V}}{\partial\lambda}, \tag{26}$$

or by (9)

$$P\dot{\Lambda} = \frac{1}{2}\frac{\mathrm{M}^2}{Q^2}\frac{\partial Q^2}{\partial\lambda} + \frac{1}{2}\frac{\mathrm{N}^2}{R^2}\frac{\partial R^2}{\partial\lambda} - \frac{1}{2}\frac{\Lambda\mathrm{M}}{PQ}\frac{\partial P^2}{\partial\mu} - \frac{1}{2}\frac{\Lambda\mathrm{N}}{PR}\frac{\partial P^2}{\partial\nu} - \frac{\partial\mathfrak{V}}{\partial\lambda}. \tag{27}$$

Similarly,

$$Q\dot{\mathrm{M}} = \frac{1}{2}\frac{\mathrm{N}^2}{R^2}\frac{\partial R^2}{\partial\mu} + \frac{1}{2}\frac{\Lambda^2}{P^2}\frac{\partial P^2}{\partial\mu} - \frac{1}{2}\frac{\mathrm{MN}}{QR}\frac{\partial Q^2}{\partial\nu} - \frac{1}{2}\frac{\mathrm{M}\Lambda}{QP}\frac{\partial Q^2}{\partial\lambda} - \frac{\partial\mathfrak{V}}{\partial\mu}, \tag{28}$$

$$R\dot{\mathrm{N}} = \frac{1}{2}\frac{\Lambda^2}{P^2}\frac{\partial P^2}{\partial\nu} + \frac{1}{2}\frac{\mathrm{M}^2}{Q^2}\frac{\partial Q^2}{\partial\nu} - \frac{1}{2}\frac{\mathrm{N}\Lambda}{RP}\frac{\partial R^2}{\partial\lambda} - \frac{1}{2}\frac{\mathrm{NM}}{RQ}\frac{\partial R^2}{\partial\mu} - \frac{\partial\mathfrak{V}}{\partial\nu}. \tag{29}$$

Substituting (27), (28), and (29) in the equation of continuity (16), we obtain

$$
\left.\begin{aligned}
&\frac{1}{P}\left\{\Lambda\frac{\partial\Psi}{\partial\lambda}+\frac{1}{2}\left[\frac{M^2}{Q^2}\frac{\partial Q^2}{\partial\lambda}+\frac{N^2}{R^2}\frac{\partial R^2}{\partial\lambda}-\frac{\Lambda M}{PQ}\frac{\partial P^2}{\partial\mu}-\frac{\Lambda N}{PR}\frac{\partial P^2}{\partial\nu}-2\frac{\partial\mathfrak{V}}{\partial\lambda}\right]\frac{\partial\Psi}{\partial\Lambda}\right\}\\
+&\frac{1}{Q}\left\{M\frac{\partial\Psi}{\partial\mu}+\frac{1}{2}\left[\frac{N^2}{R^2}\frac{\partial R^2}{\partial\mu}+\frac{\Lambda^2}{P^2}\frac{\partial P^2}{\partial\mu}-\frac{MN}{QR}\frac{\partial Q^2}{\partial\nu}-\frac{M\Lambda}{QP}\frac{\partial Q^2}{\partial\lambda}-2\frac{\partial\mathfrak{V}}{\partial\mu}\right]\frac{\partial\Psi}{\partial M}\right\}\\
+&\frac{1}{R}\left\{N\frac{\partial\Psi}{\partial\nu}+\frac{1}{2}\left[\frac{\Lambda^2}{P^2}\frac{\partial P^2}{\partial\nu}+\frac{M^2}{Q^2}\frac{\partial Q^2}{\partial\nu}-\frac{N\Lambda}{RP}\frac{\partial R^2}{\partial\lambda}-\frac{NM}{RQ}\frac{\partial R^2}{\partial\mu}-2\frac{\partial\mathfrak{V}}{\partial\nu}\right]\frac{\partial\Psi}{\partial N}\right\}=0\,.
\end{aligned}\right\}\quad(30)
$$

This is the partial differential equation for Ψ that we shall have to consider.

For a stellar system with differential motions, Ψ must have the form (cf. Eq. [17])

$$\Psi = \Psi(Q+\sigma)\,, \qquad (31)$$

where

$$
\left.\begin{aligned}
Q = a(\Lambda-\Lambda_0)^2 + b(M-M_0)^2 + c(N-N_0)^2&\\
+ 2f(M-M_0)(N-N_0) + 2g(N-N_0)(\Lambda-\Lambda_0)&\\
+ 2h(\Lambda-\Lambda_0)(M-M_0)\,.&
\end{aligned}\right\}\quad(32)
$$

We have

$$\frac{\partial\Psi}{\partial\Lambda}=\frac{d\Psi}{d(Q+\sigma)}\frac{\partial Q}{\partial\Lambda}\,;\qquad \frac{\partial\Psi}{\partial M}=\frac{d\Psi}{d(Q+\sigma)}\frac{\partial Q}{\partial M}\,;\qquad \frac{\partial\Psi}{\partial N}=\frac{d\Psi}{d(Q+\sigma)}\frac{\partial Q}{\partial N}\,. \qquad (33)$$

Similarly,

$$
\left.\begin{aligned}
\frac{\partial\Psi}{\partial\lambda}&=\frac{d\Psi}{d(Q+\sigma)}\left(\frac{\partial\sigma}{\partial\lambda}+\frac{\partial Q}{\partial\lambda}\right),\\
\frac{\partial\Psi}{\partial\mu}&=\frac{d\Psi}{d(Q+\sigma)}\left(\frac{\partial\sigma}{\partial\mu}+\frac{\partial Q}{\partial\mu}\right),\\
\frac{\partial\Psi}{\partial\nu}&=\frac{d\Psi}{d(Q+\sigma)}\left(\frac{\partial\sigma}{\partial\nu}+\frac{\partial Q}{\partial\nu}\right).
\end{aligned}\right\}\quad(34)
$$

From (32) we derive that

$$\left.\begin{aligned}\frac{\partial Q}{\partial \Lambda} &= 2a(\Lambda - \Lambda_0) + 2g(N - N_0) + 2h(M - M_0)\,,\\ \frac{\partial Q}{\partial M} &= 2b(M - M_0) + 2f(N - N_0) + 2h(\Lambda - \Lambda_0)\,,\\ \frac{\partial Q}{\partial N} &= 2c(N - N_0) + 2f(M - M_0) + 2g(\Lambda - \Lambda_0)\,.\end{aligned}\right\} \tag{35}$$

We shall introduce the quantities Δ_1, Δ_2, and Δ_3, defined by

$$\left.\begin{aligned}a\Lambda_0 + hM_0 + gN_0 &= \Delta_1\,,\\ h\Lambda_0 + bM_0 + fN_0 &= \Delta_2\,,\\ g\Lambda_0 + fM_0 + cN_0 &= \Delta_3\,.\end{aligned}\right\} \tag{36}$$

In matrix notation we can re-write (36) as

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}\begin{pmatrix} \Lambda_0 \\ M_0 \\ N_0 \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix}. \tag{37}$$

We can now write the equations (35) in the form

$$\left.\begin{aligned}\frac{\partial Q}{\partial \Lambda} &= 2(a\Lambda + hM + gN - \Delta_1)\,,\\ \frac{\partial Q}{\partial M} &= 2(h\Lambda + bM + fN - \Delta_2)\,,\\ \frac{\partial Q}{\partial N} &= 2(g\Lambda + fM + cN - \Delta_3)\,.\end{aligned}\right\} \tag{38}$$

Now,

$$\left.\begin{aligned}\frac{\partial Q}{\partial \lambda} = \Lambda^2\frac{\partial a}{\partial \lambda} + M^2\frac{\partial b}{\partial \lambda} + N^2\frac{\partial c}{\partial \lambda} + 2\left[MN\frac{\partial f}{\partial \lambda} + N\Lambda\frac{\partial g}{\partial \lambda} + \Lambda M\frac{\partial h}{\partial \lambda}\right]\\ - 2\Lambda\frac{\partial \Delta_1}{\partial \lambda} - 2M\frac{\partial \Delta_2}{\partial \lambda} - 2N\frac{\partial \Delta_3}{\partial \lambda} + \frac{\partial Q_0}{\partial \lambda}\,,\end{aligned}\right\} \tag{39}$$

where

$$Q_0 = a\Lambda_0^2 + bM_0^2 + cN_0^2 + 2fM_0N_0 + 2gN_0\Lambda_0 + 2h\Lambda_0M_0 \,. \tag{40}$$

The expressions for $\partial Q/\partial\mu$ and $\partial Q/\partial\nu$ are quite similar to (39).

We can now substitute (38) in (33), and (39) and its two analogues in (34). Finally, substituting these resulting expressions for $\partial\Psi/\partial\lambda$, etc., in (30), we obtain (after dividing throughout by $d\Psi/d(Q+\sigma)$

$$\left.\begin{aligned}
&\frac{\Lambda}{P}\left[\frac{\partial\sigma}{\partial\lambda} + \Lambda^2\frac{\partial a}{\partial\lambda} + M^2\frac{\partial b}{\partial\lambda} + N^2\frac{\partial c}{\partial\lambda} + 2MN\frac{\partial f}{\partial\lambda}\right.\\
&\quad\left. + 2N\Lambda\frac{\partial g}{\partial\lambda} + 2\Lambda M\frac{\partial h}{\partial\lambda} - 2\Lambda\frac{\partial\Delta_1}{\partial\lambda} - 2M\frac{\partial\Delta_2}{\partial\lambda} - 2N\frac{\partial\Delta_3}{\partial\lambda} + \frac{\partial Q_0}{\partial\lambda}\right]\\
&+\frac{M}{Q}\left[\frac{\partial\sigma}{\partial\mu} + \Lambda^2\frac{\partial a}{\partial\mu} + M^2\frac{\partial b}{\partial\mu} + N^2\frac{\partial c}{\partial\mu} + 2MN\frac{\partial f}{\partial\mu}\right.\\
&\quad\left. + 2N\Lambda\frac{\partial g}{\partial\mu} + 2\Lambda M\frac{\partial h}{\partial\mu} - 2\Lambda\frac{\partial\Delta_1}{\partial\mu} - 2M\frac{\partial\Delta_2}{\partial\mu} - 2N\frac{\partial\Delta_3}{\partial\mu} + \frac{\partial Q_0}{\partial\mu}\right]\\
&+\frac{N}{R}\left[\frac{\partial\sigma}{\partial\nu} + \Lambda^2\frac{\partial a}{\partial\nu} + M^2\frac{\partial b}{\partial\nu} + N^2\frac{\partial c}{\partial\nu} + 2MN\frac{\partial f}{\partial\nu}\right.\\
&\quad\left. + 2N\Lambda\frac{\partial g}{\partial\nu} + 2\Lambda M\frac{\partial h}{\partial\nu} - 2\Lambda\frac{\partial\Delta_1}{\partial\nu} - 2M\frac{\partial\Delta_2}{\partial\nu} - 2N\frac{\partial\Delta_3}{\partial\nu} + \frac{\partial Q_0}{\partial\nu}\right]\\
&+\frac{1}{P}\left[\frac{M^2}{Q^2}\frac{\partial Q^2}{\partial\lambda} + \frac{N^2}{R^2}\frac{\partial R^2}{\partial\lambda} - \frac{\Lambda M}{PQ}\frac{\partial P^2}{\partial\mu} - \frac{\Lambda N}{PR}\frac{\partial P^2}{\partial\nu} - 2\frac{\partial\mathfrak{B}}{\partial\lambda}\right][a\Lambda + hM + gN - \Delta_1]\\
&+\frac{1}{Q}\left[\frac{N^2}{R^2}\frac{\partial R^2}{\partial\mu} + \frac{\Lambda^2}{P^2}\frac{\partial P^2}{\partial\mu} - \frac{MN}{QR}\frac{\partial Q^2}{\partial\nu} - \frac{M\Lambda}{QP}\frac{\partial Q^2}{\partial\lambda} - 2\frac{\partial\mathfrak{B}}{\partial\mu}\right][h\Lambda + bM + fN - \Delta_2]\\
&+\frac{1}{R}\left[\frac{\Lambda^2}{P^2}\frac{\partial P^2}{\partial\nu} + \frac{M^2}{Q^2}\frac{\partial Q^2}{\partial\nu} - \frac{N\Lambda}{RP}\frac{\partial R^2}{\partial\lambda} - \frac{NM}{RQ}\frac{\partial R^2}{\partial\mu} - 2\frac{\partial\mathfrak{B}}{\partial\nu}\right][g\Lambda + fM + cN - \Delta_3]\\
&= 0 \,.
\end{aligned}\right\} \tag{41}$$

Equation (41) (which contains ninety-three terms) is seen to be a polynomial of the third degree in Λ, M, and N. Hence, the coefficients of the different power combinations of Λ, M, and N must vanish separately. Thus, equating the coefficients of Λ^3, M^3, N^3, Λ^2M, Λ^2N, $M^2\Lambda$, M^2N, $N^2\Lambda$, N^2M, ΛMN, Λ^2, M^2, N^2, ΛM, MN, $N\Lambda$, the constant terms, Λ, M, and N, we obtain, respectively,

$$
\left.\begin{array}{ll}
\dfrac{1}{P}\dfrac{\partial a}{\partial\lambda}+\dfrac{g}{RP^2}\dfrac{\partial P^2}{\partial\nu}+\dfrac{h}{QP^2}\dfrac{\partial P^2}{\partial\mu}=0\,, & \text{(i)}\\[2ex]
\dfrac{1}{Q}\dfrac{\partial b}{\partial\mu}+\dfrac{h}{PQ^2}\dfrac{\partial Q^2}{\partial\lambda}+\dfrac{f}{RQ^2}\dfrac{\partial Q^2}{\partial\nu}=0\,, & \text{(ii)}\\[2ex]
\dfrac{1}{R}\dfrac{\partial c}{\partial\nu}+\dfrac{f}{QR^2}\dfrac{\partial R^2}{\partial\mu}+\dfrac{g}{PR^2}\dfrac{\partial R^2}{\partial\lambda}=0\,, & \text{(iii)}\\[2ex]
\dfrac{2}{P}\dfrac{\partial h}{\partial\lambda}+\dfrac{1}{Q}\dfrac{\partial a}{\partial\mu}-(a-b)\dfrac{1}{QP^2}\dfrac{\partial P^2}{\partial\mu}+\dfrac{f}{RP^2}\dfrac{\partial P^2}{\partial\nu}-\dfrac{h}{PQ^2}\dfrac{\partial Q^2}{\partial\lambda}=0\,, & \text{(iv)}\\[2ex]
\dfrac{2}{P}\dfrac{\partial g}{\partial\lambda}+\dfrac{1}{R}\dfrac{\partial a}{\partial\nu}-(a-c)\dfrac{1}{RP^2}\dfrac{\partial P^2}{\partial\nu}+\dfrac{f}{QP^2}\dfrac{\partial P^2}{\partial\mu}-\dfrac{g}{PR^2}\dfrac{\partial R^2}{\partial\lambda}=0\,, & \text{(v)}\\[2ex]
\dfrac{2}{Q}\dfrac{\partial h}{\partial\mu}+\dfrac{1}{P}\dfrac{\partial b}{\partial\lambda}-(b-a)\dfrac{1}{PQ^2}\dfrac{\partial Q^2}{\partial\lambda}+\dfrac{g}{RQ^2}\dfrac{\partial Q^2}{\partial\nu}-\dfrac{h}{QP^2}\dfrac{\partial P^2}{\partial\mu}=0\,, & \text{(vi)}\\[2ex]
\dfrac{2}{Q}\dfrac{\partial f}{\partial\mu}+\dfrac{1}{R}\dfrac{\partial b}{\partial\nu}-(b-c)\dfrac{1}{RQ^2}\dfrac{\partial Q^2}{\partial\nu}+\dfrac{g}{PQ^2}\dfrac{\partial Q^2}{\partial\lambda}-\dfrac{f}{QR^2}\dfrac{\partial R^2}{\partial\mu}=0\,, & \text{(vii)}\\[2ex]
\dfrac{2}{R}\dfrac{\partial g}{\partial\nu}+\dfrac{1}{P}\dfrac{\partial c}{\partial\lambda}-(c-a)\dfrac{1}{PR^2}\dfrac{\partial R^2}{\partial\lambda}+\dfrac{h}{QR^2}\dfrac{\partial R^2}{\partial\mu}-\dfrac{g}{RP^2}\dfrac{\partial P^2}{\partial\nu}=0\,, & \text{(viii)}\\[2ex]
\dfrac{2}{R}\dfrac{\partial f}{\partial\nu}+\dfrac{1}{Q}\dfrac{\partial c}{\partial\mu}-(c-b)\dfrac{1}{QR^2}\dfrac{\partial R^2}{\partial\mu}+\dfrac{h}{PR^2}\dfrac{\partial R^2}{\partial\lambda}-\dfrac{f}{RQ^2}\dfrac{\partial Q^2}{\partial\nu}=0\,, & \text{(ix)}\\[2ex]
\left.\begin{array}{l}
\dfrac{g}{QP^2}\dfrac{\partial P^2}{\partial\mu}+\dfrac{h}{RP^2}\dfrac{\partial P^2}{\partial\nu}+\dfrac{h}{RQ^2}\dfrac{\partial Q^2}{\partial\nu}+\dfrac{f}{PQ^2}\dfrac{\partial Q^2}{\partial\lambda}+\dfrac{f}{PR^2}\dfrac{\partial R^2}{\partial\lambda}\\[2ex]
\qquad\qquad+\dfrac{g}{QR^2}\dfrac{\partial R^2}{\partial\mu}-2\left(\dfrac{1}{P}\dfrac{\partial f}{\partial\lambda}+\dfrac{1}{Q}\dfrac{\partial g}{\partial\mu}+\dfrac{1}{R}\dfrac{\partial h}{\partial\nu}\right)=0\,,
\end{array}\right\} & \text{(x)}
\end{array}\right\}\text{(I)}
$$

$$
\left.\begin{array}{ll}
\dfrac{2}{P}\dfrac{\partial\Delta_1}{\partial\lambda}+\dfrac{\Delta_2}{QP^2}\dfrac{\partial P^2}{\partial\mu}+\dfrac{\Delta_3}{RP^2}\dfrac{\partial P^2}{\partial\nu}=0\,, & \text{(i)}\\[2ex]
\dfrac{2}{Q}\dfrac{\partial\Delta_2}{\partial\mu}+\dfrac{\Delta_3}{RQ^2}\dfrac{\partial Q^2}{\partial\nu}+\dfrac{\Delta_1}{PQ^2}\dfrac{\partial Q^2}{\partial\lambda}=0\,, & \text{(ii)}\\[2ex]
\dfrac{2}{R}\dfrac{\partial\Delta_3}{\partial\nu}+\dfrac{\Delta_1}{PR^2}\dfrac{\partial R^2}{\partial\lambda}+\dfrac{\Delta_2}{QR^2}\dfrac{\partial R^2}{\partial\mu}=0\,, & \text{(iii)}\\[2ex]
\dfrac{2}{P}\dfrac{\partial\Delta_2}{\partial\lambda}+\dfrac{2}{Q}\dfrac{\partial\Delta_1}{\partial\mu}-\dfrac{\Delta_1}{QP^2}\dfrac{\partial P^2}{\partial\mu}-\dfrac{\Delta_2}{PQ^2}\dfrac{\partial Q^2}{\partial\lambda}=0\,, & \text{(iv)}\\[2ex]
\dfrac{2}{Q}\dfrac{\partial\Delta_3}{\partial\mu}+\dfrac{2}{R}\dfrac{\partial\Delta_2}{\partial\nu}-\dfrac{\Delta_2}{RQ^2}\dfrac{\partial Q^2}{\partial\nu}-\dfrac{\Delta_3}{QR^2}\dfrac{\partial R^2}{\partial\mu}=0\,, & \text{(v)}\\[2ex]
\dfrac{2}{R}\dfrac{\partial\Delta_1}{\partial\nu}+\dfrac{2}{P}\dfrac{\partial\Delta_3}{\partial\lambda}-\dfrac{\Delta_3}{PR^2}\dfrac{\partial R^2}{\partial\lambda}-\dfrac{\Delta_1}{RP^2}\dfrac{\partial P^2}{\partial\nu}=0\,, & \text{(vi)}
\end{array}\right\}\text{(II)}
$$

$$\frac{\Delta_1}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{\Delta_2}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}+\frac{\Delta_3}{R}\frac{\partial\mathfrak{B}}{\partial\nu}=0\,, \tag{III}$$

and

$$\left.\begin{aligned}
&\frac{1}{P}\frac{\partial}{\partial\lambda}(\sigma+Q_0)-2\left(\frac{a}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{h}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}+\frac{g}{R}\frac{\partial\mathfrak{B}}{\partial\nu}\right)=0\,, &\text{(i)}\\
&\frac{1}{Q}\frac{\partial}{\partial\mu}(\sigma+Q_0)-2\left(\frac{h}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{b}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}+\frac{f}{R}\frac{\partial\mathfrak{B}}{\partial\nu}\right)=0\,, &\text{(ii)}\\
&\frac{1}{R}\frac{\partial}{\partial\nu}(\sigma+Q_0)-2\left(\frac{g}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{f}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}+\frac{c}{R}\frac{\partial\mathfrak{B}}{\partial\nu}\right)=0\,. &\text{(iii)}
\end{aligned}\right\} \tag{IV}$$

We see that the twenty partial differential equations which result break up into four distinct sets of equations. The first group of ten equations involve only the coefficients of the ellipsoid; and, as we shall see, these differential equations are sufficient to determine them uniquely apart from integration constants. The second group of six equations involve only the Δ's and can thus be treated independently of the others. It may be recalled that, according to our definition (Eq. [37]), we have

$$\begin{pmatrix} a & h & g\\ h & b & f\\ g & f & c\end{pmatrix}\begin{pmatrix}\Lambda_0\\ M_0\\ N_0\end{pmatrix}=\begin{pmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\end{pmatrix}. \tag{42}$$

The solution of the second set of equations (II) will determine the motions (Λ_0, M_0, N_0) of the local centroids.

The third equation (III) is seen to be a linear homogeneous partial differential equation of the Lagrangian type for $\mathfrak{B}$. From this equation we infer that

$$\mathfrak{B}(\lambda,\mu,\nu)\equiv\mathfrak{B}(I_1, I_2)\,, \tag{43}$$

where I_1 and I_2 are any two independent integrals of the corresponding subsidiary equations

$$\frac{Pd\lambda}{\Delta_1}=\frac{Qd\mu}{\Delta_2}=\frac{Rd\nu}{\Delta_3}\,. \tag{IIIa}$$

The last group of three equations (IV) is of a nature different from the rest; and for reasons which will become apparent later, we

shall refer to these three equations as the "compatibility conditions." The equations (IV) can be simplified by introducing a functions χ defined by

$$-\chi = \sigma + Q_0 \,. \tag{44}$$

The compatibility conditions can now be written as a single matrix equation in the form

$$2\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}\begin{pmatrix} \dfrac{1}{P}\dfrac{\partial\mathfrak{B}}{\partial\lambda} \\[2ex] \dfrac{1}{Q}\dfrac{\partial\mathfrak{B}}{\partial\nu} \\[2ex] \dfrac{1}{R}\dfrac{\partial\mathfrak{B}}{\partial\mu} \end{pmatrix} + \begin{pmatrix} \dfrac{1}{P}\dfrac{\partial\chi}{\partial\lambda} \\[2ex] \dfrac{1}{Q}\dfrac{\partial\chi}{\partial\mu} \\[2ex] \dfrac{1}{R}\dfrac{\partial\chi}{\partial\nu} \end{pmatrix} = 0 \,. \tag{IVa}$$

If we multiply the foregoing equation on the left by the vector $(\mathrm{\Lambda}_0, \mathrm{M}_0, \mathrm{N}_0)$, now regarding it as a matrix of just one row, we obtain (on using [42])

$$2(\Delta_1\ \Delta_2\ \Delta_3)\begin{pmatrix} \dfrac{1}{P}\dfrac{\partial\mathfrak{B}}{\partial\lambda} \\[2ex] \dfrac{1}{Q}\dfrac{\partial\mathfrak{B}}{\partial\mu} \\[2ex] \dfrac{1}{R}\dfrac{\partial\mathfrak{B}}{\partial\nu} \end{pmatrix} + (\mathrm{\Lambda}_0\ \mathrm{M}_0\ \mathrm{N}_0)\begin{pmatrix} \dfrac{1}{P}\dfrac{\partial\chi}{\partial\lambda} \\[2ex] \dfrac{1}{Q}\dfrac{\partial\chi}{\partial\mu} \\[2ex] \dfrac{1}{R}\dfrac{\partial\chi}{\partial\nu} \end{pmatrix} = 0 \,. \tag{45}$$

From (III) and (45) we now obtain a homogeneous linear partial differential equation of the Lagrangian type for χ:

$$\frac{\mathrm{\Lambda}_0}{P}\frac{\partial\chi}{\partial\lambda} + \frac{\mathrm{M}_0}{Q}\frac{\partial\chi}{\partial\mu} + \frac{\mathrm{N}_0}{R}\frac{\partial\chi}{\partial\nu} = 0 \,. \tag{IVb}$$

From (IV*b*) we infer that

$$\chi \equiv \chi(J_1, J_2) \,, \tag{46}$$

where J_1 and J_2 are any two independent integrals of the subsidiary equations

$$\frac{Pd\lambda}{\mathrm{\Lambda}_0} = \frac{Qd\mu}{\mathrm{M}_0} = \frac{Rd\nu}{\mathrm{N}_0} \,. \tag{IVc}$$

Our fundamental differential equations (I), (II), (III), and (IV) take more symmetrical forms if we introduce the quantities A, B, C, F, G, H, ξ, η, and ζ, defined as follows:

$$\left.\begin{aligned} a &= P^2A\ ; & b &= Q^2B\ ; & c &= R^2C\ , \\ f &= QRF\ ; & g &= RPG\ ; & h &= PQH\ , \end{aligned}\right\} \tag{47}$$

and

$$\Delta_1 = \xi P\ ; \qquad \Delta_2 = \eta Q\ ; \qquad \Delta_3 = \zeta R\ . \tag{48}$$

Among the variables A, B, C, F, G, and H and ξ, η, and ζ, we have the relation (cf. Eq. [37])

$$\begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix}\begin{pmatrix} P\Lambda_0 \\ QM_0 \\ RN_0 \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}. \tag{48$'$}$$

In terms of these new variables the differential equations are

$$\left.\begin{aligned}
&\frac{\partial A}{\partial\lambda} + 2\left(A\,\frac{\partial \log P}{\partial\lambda} + H\,\frac{\partial \log P}{\partial\mu} + G\,\frac{\partial \log P}{\partial\nu}\right) = 0 && \text{(i)}\\
&\frac{\partial B}{\partial\mu} + 2\left(H\,\frac{\partial \log Q}{\partial\lambda} + B\,\frac{\partial \log Q}{\partial\mu} + F\,\frac{\partial \log Q}{\partial\nu}\right) = 0\ , && \text{(ii)}\\
&\frac{\partial C}{\partial\nu} + 2\left(G\,\frac{\partial \log R}{\partial\lambda} + F\,\frac{\partial \log R}{\partial\mu} + C\,\frac{\partial \log R}{\partial\nu}\right) = 0\ , && \text{(iii)}\\
&\frac{P^2}{Q^2}\frac{\partial A}{\partial\mu} + 2\,\frac{\partial H}{\partial\lambda} + 2\left(H\,\frac{\partial \log P}{\partial\lambda} + B\,\frac{\partial \log P}{\partial\mu} + F\,\frac{\partial \log P}{\partial\nu}\right) = 0\ , && \text{(iv)}\\
&\frac{P^2}{R^2}\frac{\partial A}{\partial\nu} + 2\,\frac{\partial G}{\partial\lambda} + 2\left(G\,\frac{\partial \log P}{\partial\lambda} + F\,\frac{\partial \log P}{\partial\mu} + C\,\frac{\partial \log P}{\partial\nu}\right) = 0\ , && \text{(v)}\\
&\frac{Q^2}{P^2}\frac{\partial B}{\partial\lambda} + 2\,\frac{\partial H}{\partial\mu} + 2\left(A\,\frac{\partial \log Q}{\partial\lambda} + H\,\frac{\partial \log Q}{\partial\mu} + G\,\frac{\partial \log Q}{\partial\nu}\right) = 0\ , && \text{(vi)}\\
&\frac{Q^2}{R^2}\frac{\partial B}{\partial\nu} + 2\,\frac{\partial F}{\partial\mu} + 2\left(G\,\frac{\partial \log Q}{\partial\lambda} + F\,\frac{\partial \log Q}{\partial\mu} + C\,\frac{\partial \log Q}{\partial\nu}\right) = 0\ , && \text{(vii)}\\
&\frac{R^2}{P^2}\frac{\partial C}{\partial\lambda} + 2\,\frac{\partial G}{\partial\nu} + 2\left(A\,\frac{\partial \log R}{\partial\lambda} + H\,\frac{\partial \log R}{\partial\mu} + G\,\frac{\partial \log R}{\partial\nu}\right) = 0\ , && \text{(viii)}\\
&\frac{R^2}{Q^2}\frac{\partial C}{\partial\mu} + 2\,\frac{\partial F}{\partial\nu} + 2\left(H\,\frac{\partial \log R}{\partial\lambda} + B\,\frac{\partial \log R}{\partial\mu} + F\,\frac{\partial \log R}{\partial\nu}\right) = 0\ , && \text{(ix)}\\
&\frac{1}{P^2}\frac{\partial F}{\partial\lambda} + \frac{1}{Q^2}\frac{\partial G}{\partial\mu} + \frac{1}{R^2}\frac{\partial H}{\partial\nu} = 0\ , && \text{(x)}
\end{aligned}\right\} \tag{I$_1$}$$

$$\left.\begin{aligned}
&\frac{\partial\xi}{\partial\lambda}+\xi\,\frac{\partial\log P}{\partial\lambda}+\eta\,\frac{\partial\log P}{\partial\mu}+\zeta\,\frac{\partial\log P}{\partial\nu}=0\,, &&\text{(i)}\\
&\frac{\partial\eta}{\partial\mu}+\xi\,\frac{\partial\log Q}{\partial\lambda}+\eta\,\frac{\partial\log Q}{\partial\mu}+\zeta\,\frac{\partial\log Q}{\partial\nu}=0\,, &&\text{(ii)}\\
&\frac{\partial\zeta}{\partial\nu}+\xi\,\frac{\partial\log R}{\partial\lambda}+\eta\,\frac{\partial\log R}{\partial\mu}+\zeta\,\frac{\partial\log R}{\partial\nu}=0\,, &&\text{(iii)}\\
&P^2\,\frac{\partial\xi}{\partial\mu}+Q^2\,\frac{\partial\eta}{\partial\lambda}=0\,, &&\text{(iv)}\\
&Q^2\,\frac{\partial\eta}{\partial\nu}+R^2\,\frac{\partial\zeta}{\partial\mu}=0\,, &&\text{(v)}\\
&R^2\,\frac{\partial\zeta}{\partial\lambda}+P^2\,\frac{\partial\xi}{\partial\nu}=0\,, &&\text{(vi)}
\end{aligned}\right\}\qquad(\mathrm{II}_1)$$

$$\xi\,\frac{\partial\mathfrak{B}}{\partial\lambda}+\eta\,\frac{\partial\mathfrak{B}}{\partial\mu}+\zeta\,\frac{\partial\mathfrak{B}}{\partial\nu}=0\,,\qquad(\mathrm{III}_1)$$

and

$$\left.\begin{aligned}
&\frac{\partial\chi}{\partial\lambda}+2P^2\left(A\,\frac{\partial\mathfrak{B}}{\partial\lambda}+H\,\frac{\partial\mathfrak{B}}{\partial\mu}+G\,\frac{\partial\mathfrak{B}}{\partial\nu}\right)=0\,, &&\text{(i)}\\
&\frac{\partial\chi}{\partial\mu}+2Q^2\left(H\,\frac{\partial\mathfrak{B}}{\partial\lambda}+B\,\frac{\partial\mathfrak{B}}{\partial\mu}+F\,\frac{\partial\mathfrak{B}}{\partial\nu}\right)=0\,, &&\text{(ii)}\\
&\frac{\partial\chi}{\partial\nu}+2R^2\left(G\,\frac{\partial\mathfrak{B}}{\partial\lambda}+F\,\frac{\partial\mathfrak{B}}{\partial\mu}+C\,\frac{\partial\mathfrak{B}}{\partial\nu}\right)=0\,. &&\text{(iii)}
\end{aligned}\right\}\qquad(\mathrm{IV}_1)$$

From (III_1) we now infer that

$$\mathfrak{B}(\lambda,\mu,\nu)\equiv\mathfrak{B}(I_1,I_2)\,,\qquad(49)$$

where I_1 and I_2 are the two first integrals of the subsidiary equations

$$\frac{d\lambda}{\xi}=\frac{d\mu}{\eta}=\frac{d\nu}{\zeta}\,.\qquad(\mathrm{III}_1a)$$

4. *The equations for the two-dimensional problem.*—It is often possible to obtain an insight into the essential features of a situation by considering the two-dimensional problem. In this manner we can, in the first instance, avoid the additional complexities involved in dealing with the general three-dimensional problem. We shall therefore write down our fundamental equations for the two-dimensional case.

Let λ and μ define two orthogonal families of curves in a plane. Then λ and μ will suffice as a basis for an orthogonal system of coordinates. Let $Pd\lambda$ and $Qd\mu$ be the lengths of the elementary arcs along the principal directions at (λ, μ). Then the components of the velocity Λ and M along the principal directions at (λ, μ) are

$$\Lambda = P\dot{\lambda} \; ; \qquad \mathrm{M} = Q\dot{\mu} \, . \tag{50}$$

Our assumption now regarding the distribution function Ψ is

$$\Psi(\lambda, \mu) = \Psi(Q + \sigma[\lambda, \mu]) \, , \tag{51}$$

where

$$Q = a(\Lambda - \Lambda_0)^2 + b(\mathrm{M} - \mathrm{M}_0)^2 + 2h(\Lambda - \Lambda_0)(\mathrm{M} - \mathrm{M}_0) \, . \tag{52}$$

In (52), Λ_0 and M_0 are the components of the motion of the local centroid at (λ, μ) along the principal directions at (λ, μ).

As before, let us introduce the quantities Δ_1, Δ_2, Q_0, and χ defined by

$$\Delta_1 = a\Lambda_0 + h\mathrm{M}_0 \; ; \qquad \Delta_2 = h\Lambda_0 + b\mathrm{M}_0 \, , \tag{53}$$

$$Q_0 = a\Lambda_0^2 + 2h\Lambda_0\mathrm{M}_0 + b\mathrm{M}_0^2 \, , \tag{54}$$

and

$$-\chi = \sigma + Q_0 \, . \tag{55}$$

Our fundamental equation can now be obtained from (I), (II), (III), and (IV) by setting $c = f = g = 0$ and further ignoring all the terms involving differentiations with respect to ν. We have

$$\left.\begin{aligned}
&\frac{1}{P}\frac{\partial a}{\partial \lambda} + \frac{h}{QP^2}\frac{\partial P^2}{\partial \mu} = 0 \, , && \text{(i)}\\
&\frac{1}{Q}\frac{\partial b}{\partial \mu} + \frac{h}{PQ^2}\frac{\partial Q^2}{\partial \lambda} = 0 \, , && \text{(ii)}\\
&\frac{2}{P}\frac{\partial h}{\partial \lambda} + \frac{1}{Q}\frac{\partial a}{\partial \mu} - (a - b)\frac{1}{QP^2}\frac{\partial P^2}{\partial \mu} - \frac{h}{PQ^2}\frac{\partial Q^2}{\partial \lambda} = 0 \, , && \text{(iii)}\\
&\frac{2}{Q}\frac{\partial h}{\partial \mu} + \frac{1}{P}\frac{\partial b}{\partial \lambda} - (b - a)\frac{1}{PQ^2}\frac{\partial Q^2}{\partial \lambda} - \frac{h}{QP^2}\frac{\partial P^2}{\partial \mu} = 0 \, , && \text{(iv)}
\end{aligned}\right\} \quad (\mathrm{I}_2)$$

$$\left.\begin{aligned}
&\frac{2}{P}\frac{\partial\Delta_1}{\partial\lambda}+\frac{\Delta_2}{QP^2}\frac{\partial P^2}{\partial\mu}=0\,, && \text{(i)}\\
&\frac{2}{Q}\frac{\partial\Delta_2}{\partial\mu}+\frac{\Delta_1}{PQ^2}\frac{\partial Q^2}{\partial\lambda}=0\,, && \text{(ii)}\\
&\frac{2}{P}\frac{\partial\Delta_2}{\partial\lambda}+\frac{2}{Q}\frac{\partial\Delta_1}{\partial\mu}-\frac{\Delta_1}{QP^2}\frac{\partial P^2}{\partial\mu}-\frac{\Delta_2}{PQ^2}\frac{\partial Q^2}{\partial\lambda}=0\,, && \text{(iii)}
\end{aligned}\right\}\qquad(\mathrm{II}_2)$$

$$\frac{\Delta_1}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{\Delta_2}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}=0\,,\qquad(\mathrm{III}_2)$$

and

$$\left.\begin{aligned}
&\frac{1}{P}\frac{\partial\chi}{\partial\lambda}+2\left(\frac{a}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{h}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}\right)=0\,, && \text{(i)}\\
&\frac{1}{Q}\frac{\partial\chi}{\partial\mu}+2\left(\frac{h}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{b}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}\right)=0\,, && \text{(ii)}
\end{aligned}\right\}\qquad(\mathrm{IV}_2)$$

To solve the dynamical problem formulated in § 2 we have to discuss the methods of solving the fundamental partial differential equations (I), (II), (III), and (IV), which are twenty in number for the general problem and ten in number for the two-dimensional problem. We proceed now to this discussion.

II. THE TWO-DIMENSIONAL PROBLEM

5. *The equations in Cartesian co-ordinates.*—For the fundamental frame of reference we shall choose a rectangular system of co-ordinates (x, y). For such a system $P = 1$ and $Q = 1$, and the equations (I_2), (II_2), (III_2), and (IV_2) now become

$$\left.\begin{aligned}
&\frac{\partial a}{\partial x}=0\,,\quad\text{(i)};\qquad 2\frac{\partial h}{\partial x}+\frac{\partial a}{\partial y}=0\,,\quad\text{(iii)}\\
&\frac{\partial b}{\partial y}=0\,,\quad\text{(ii)};\qquad 2\frac{\partial h}{\partial y}+\frac{\partial b}{\partial x}=0\,,\quad\text{(iv)}
\end{aligned}\right\}\qquad(\mathrm{I}_3)$$

$$\frac{\partial\Delta_1}{\partial x}=0\,,\quad\text{(i)};\qquad\frac{\partial\Delta_2}{\partial y}=0\,,\quad\text{(ii)};\qquad\frac{\partial\Delta_1}{\partial y}+\frac{\partial\Delta_2}{\partial x}=0\,,\quad\text{(iii)},\qquad(\mathrm{II}_3)$$

$$\Delta_1\frac{\partial\mathfrak{B}}{\partial x}+\Delta_2\frac{\partial\mathfrak{B}}{\partial y}=0\,,\qquad(\mathrm{III}_3)$$

and

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix} \operatorname{grad} \mathfrak{V} = -\tfrac{1}{2} \operatorname{grad} \chi . \tag{IV$_3$}$$

6. *The solution for the coefficients of the velocity ellipse.*—We now consider the four equations (I_3). From (i) and (ii) we infer that a and b are independent of x and y, respectively. On the other hand, from (i) and (iii) and similarly from (ii) and (iv) we obtain

$$\frac{\partial^2 h}{\partial x^2} = 0 ; \qquad \frac{\partial^2 h}{\partial y^2} = 0 ; \tag{56}$$

in other words, h is linear both in x and y. Consequently, it should be of the form

$$h = h_1 + h_2 x + h_3 y + h_4 xy , \tag{57}$$

where $h_1, \ldots , h_4$ are constants. From (iii) we now obtain

$$\frac{\partial a}{\partial y} = -2h_2 - 2h_4 y , \tag{58}$$

or, integrating,

$$a = -2h_2 y - h_4 y^2 - a_0 , \tag{59}$$

where a_0 is a constant of integration. Similarly from (iv) we derive

$$b = -2h_3 x - h_4 x^2 - b_0 , \tag{60}$$

where b_0 is another constant.

7. *The solution for the motions of the local centroids.*—We now consider the three equations (II_3). From (i) and (ii) we see that Δ_1 and Δ_2 are independent of x and y, respectively. On the other hand, by partially differentiating (3), first with respect to x and then with respect to y, we obtain

$$\frac{\partial^2 \Delta_2}{\partial x^2} = 0 ; \qquad \frac{\partial^2 \Delta_1}{\partial y^2} = 0 ; \tag{61}$$

i.e., Δ_2 and Δ_1 are linear in x and y, respectively. The solution is now seen to be

$$\Delta_1 = \beta y + \delta_1 \; ; \qquad \Delta_2 = -\beta x + \delta_2 \,, \tag{62}$$

where β, δ_1, and δ_2 are arbitrary constants.

8. *The solution of the partial differential equation (III_3) for $\mathfrak{B}$.*—From (III_3) we infer that

$$\mathfrak{B}(x, y) \equiv \mathfrak{B}(I) \,, \tag{63}$$

where I denotes the first integral of the subsidiary equation

$$\frac{dx}{\Delta_1} = \frac{dy}{\Delta_2} \,, \tag{64}$$

or, according to our solution (62) for the Δ's,

$$\frac{dx}{\beta y + \delta_1} = \frac{dy}{-\beta x + \delta_2} \,. \tag{65}$$

We can re-write (65) in the form

$$(\beta x - \delta_2)dx + (\beta y + \delta_1)dy = 0 \,. \tag{66}$$

Equation (66) integrates out immediately. We have

$$\beta(x^2 + y^2) + 2\delta_1 y - 2\delta_2 x = \text{constant} \,. \tag{67}$$

Case (*i*): $\beta \neq 0$.—In this case we can write, instead of (67),

$$\left(x - \frac{\delta_2}{\beta}\right)^2 + \left(y + \frac{\delta_1}{\beta}\right)^2 = \text{constant} \,. \tag{68}$$

Hence, by (63)

$$\mathfrak{B} \equiv \mathfrak{B}\left\{ \left(x - \frac{\delta_2}{\beta}\right)^2 + \left(y + \frac{\delta_1}{\beta}\right)^2 \right\} . \tag{69}$$

In other words, $\mathfrak{B}$ *has a circular symmetry about the point* (δ_2/β, $-\delta_1/\beta$).

Case (ii): $\beta = 0$.—We now have

$$\delta_1 y - \delta_2 x = \text{constant} . \tag{70}$$

Hence,

$$\mathfrak{V} = \mathfrak{V}(\delta_1 y - \delta_2 x) . \tag{71}$$

In other words, $\mathfrak{V}$ *is constant along lines parallel to*

$$\frac{x}{\delta_1} = \frac{y}{\delta_2} . \tag{72}$$

Case (iii): $\beta = \delta_1 = \delta_2 = 0$.—In this case $\Delta_1 = 0$ and $\Delta_2 = 0$, and there is no restriction on $\mathfrak{V}$. On the other hand, since $\Delta_1 = \Delta_2 = 0$, we should have (cf. Eq. [53])

$$\left.\begin{aligned} aU_0 + hV_0 &= 0 , \\ hU_0 + bV_0 &= 0 . \end{aligned}\right\} \tag{73}$$

From (73) we now conclude that either $U_0 = V_0 = 0$ or $ab - h^2 = 0$. In the latter case

$$\frac{U_0}{V_0} = -\frac{h}{a} = -\frac{b}{h} . \tag{74}$$

We postpone to § 11 the consideration of the physical consequences arising from the condition $ab - h^2 = 0$ and also equation (74). Meanwhile, we can summarize the conclusions reached from the discussion of equation (III_3) as follows:

For a stellar system in two dimensions with differential motions either (1) the equipotential lines are a set of concentric circles (or a set of parallel lines in the limiting case) or (2) the motions of the local centroids are everywhere zero or (3) $\text{ab} - \text{h}^2 = 0$.

9. *The discussion of the compatibility conditions (the case of circular symmetry).*—In § 8 we showed that in the general case (i.e., when $\beta \neq 0$) the potential function $\mathfrak{V}$ must have a circular symmetry about the point $(\delta_2/\beta, -\delta_1/\beta)$. By a parallel translation of the fundamental frame of reference we can arrange to have the origin of

the system of co-ordinates coincide with the center of symmetry of $\mathfrak{B}$. We shall assume that this has been done, so that now

$$\mathfrak{B} \equiv \mathfrak{B}(x^2 + y^2) . \tag{75}$$

Now the compatibility conditions (IV_3) can be written as

$$a \frac{\partial \mathfrak{B}}{\partial x} + h \frac{\partial \mathfrak{B}}{\partial y} = -\frac{1}{2} \frac{\partial \chi}{\partial x} , \tag{76}$$

$$h \frac{\partial \mathfrak{B}}{\partial x} + b \frac{\partial \mathfrak{B}}{\partial y} = -\frac{1}{2} \frac{\partial \chi}{\partial y} . \tag{77}$$

In order that the two equations may be compatible (or integrable), we should have

$$\frac{\partial}{\partial y}\left(a \frac{\partial \mathfrak{B}}{\partial x} + h \frac{\partial \mathfrak{B}}{\partial y}\right) = \frac{\partial}{\partial x}\left(h \frac{\partial \mathfrak{B}}{\partial x} + b \frac{\partial \mathfrak{B}}{\partial y}\right) , \tag{78}$$

or, after some reductions,

$$\left.\begin{aligned} \left(\frac{\partial a}{\partial y} - \frac{\partial h}{\partial x}\right)\frac{\partial \mathfrak{B}}{\partial x} + \left(\frac{\partial h}{\partial y} - \frac{\partial b}{\partial x}\right)\frac{\partial \mathfrak{B}}{\partial y} + (a - b) \frac{\partial^2 \mathfrak{B}}{\partial x \partial y} \\ + h \left(\frac{\partial^2 \mathfrak{B}}{\partial y^2} - \frac{\partial^2 \mathfrak{B}}{\partial x^2}\right) = 0 . \end{aligned}\right\} \tag{79}$$

Let us introduce a new variable, τ, defined by

$$\tau = \tfrac{1}{2}(x^2 + y^2) . \tag{80}$$

Then, since $\mathfrak{B} \equiv \mathfrak{B}(\tau)$, we have

$$\left.\begin{aligned} \frac{\partial \mathfrak{B}}{\partial x} &= \frac{d\mathfrak{B}}{d\tau} \frac{\partial \tau}{\partial x} = x \frac{d\mathfrak{B}}{d\tau} , \\ \frac{\partial \mathfrak{B}}{\partial y} &= \frac{d\mathfrak{B}}{d\tau} \frac{\partial \tau}{\partial y} = y \frac{d\mathfrak{B}}{d\tau} . \end{aligned}\right\} \tag{81}$$

Further, from (81) we easily find that

$$\left.\begin{aligned}\frac{\partial^2\mathfrak{B}}{\partial x\partial y} &= xy\,\frac{d^2\mathfrak{B}}{d\tau^2}\,,\\ \frac{\partial^2\mathfrak{B}}{\partial x^2} &= \frac{d\mathfrak{B}}{d\tau} + x^2\,\frac{d^2\mathfrak{B}}{d\tau^2}\,,\\ \frac{\partial^2\mathfrak{B}}{\partial y^2} &= \frac{d\mathfrak{B}}{d\tau} + y^2\,\frac{d^2\mathfrak{B}}{d\tau^2}\,.\end{aligned}\right\} \tag{82}$$

Substituting (81) and (82) in (79), we obtain, after some minor transformations,

$$\frac{d^2\mathfrak{B}}{d\tau^2}\Big/\frac{d\mathfrak{B}}{d\tau} = \frac{x\left(\frac{\partial a}{\partial y} - \frac{\partial h}{\partial x}\right) + y\left(\frac{\partial h}{\partial y} - \frac{\partial b}{\partial x}\right)}{(b-a)xy + h(x^2 - y^2)}\,. \tag{83}$$

If we substitute in (83) for a, b, and h, according to equations (59), (60), and (57), we obtain

$$\frac{d^2\mathfrak{B}}{d\tau^2}\Big/\frac{d\mathfrak{B}}{d\tau} = S\,, \tag{84}$$

where

$$S = 3\,\frac{h_3 y - h_2 x}{h(x^2 - y^2) + xy(b-a)}\,. \tag{85}$$

We now see that, if the compatibility conditions are to be satisfied identically, then S should take the indeterminate form 0/0. Otherwise we should insist that S should be a function of τ only, since the left-hand side of (84) is a function only of τ. This latter condition is equivalent to restricting S such that

$$y\,\frac{\partial S}{\partial x} - x\,\frac{\partial S}{\partial y} = 0\,. \tag{86}$$

The importance of satisfying the compatibility conditions identically is that these conditions do not then require any further restrictions on the form of $\mathfrak{B}$ other than those required by equation (III)—in this case the requirement of a circular symmetry for $\mathfrak{B}$. If the compatibility conditions are not satisfied identically, then

(86) should be satisfied, and this will require, as we shall see, only certain special forms for $\mathfrak{B}(\tau)$.

Case 1: S *indeterminate.*—By (85) we should now have

$$h_3 y - h_2 x = 0 \tag{87}$$

and

$$h(x^2 - y^2) + xy(b - a) = 0\,. \tag{88}$$

Equation (87) implies that $h_2 = h_3 = 0$. Hence, our expressions for a, b, and h now reduce to (cf. Eqs. [57], [59], [60])

$$a = -h_4 y^2 - a_0\,; \qquad b = -h_4 x^2 - b_0 \tag{89}$$

and

$$h = h_1 + h_4 xy\,. \tag{90}$$

Substituting (89) and (90) in (88), we have

$$(h_1 + h_4 xy)(x^2 - y^2) + xy(-b_0 - h_4 x^2 + h_4 y^2 + a_0) = 0\,. \tag{91}$$

The terms in h_4 cancel identically, and we are left with

$$h_1(x^2 - y^2) + xy(a_0 - b_0) = 0\,. \tag{92}$$

Hence,

$$h_1 = 0\,, \qquad a_0 = b_0 = -\kappa_1 \text{ (say)}\,. \tag{93}$$

We thus see that, if the compatibility conditions are satisfied identically, the solutions for the coefficients of the ellipse can involve only two arbitrary constants, as against six in the general solution obtained in § 6. If we now set $-h_4 = \kappa_2$, then we have

$$\left.\begin{aligned} a &= \kappa_2 y^2 + \kappa_1\,, \\ b &= \kappa_2 x^2 + \kappa_1\,, \\ h &= -\kappa_2 xy\,. \end{aligned}\right\} \tag{94}$$

If u and v are the residual velocities, then

$$u = U - U_0\,; \qquad v = V - V_0\,. \tag{95}$$

The fundamental quadratic form is now given by

$$Q = (\kappa_2 y^2 + \kappa_1)u^2 + (\kappa_2 x^2 + \kappa_1)v^2 - 2\kappa_2 xyuv\,. \tag{96}$$

If ϵ is the angle which a principal axis to the ellipse Q = constant makes with the positive x-direction, then, according to a well-known formula,

$$\tan 2\epsilon = \frac{2h}{a-b}\,, \tag{97}$$

or from (94)

$$\tan 2\epsilon = \frac{2xy}{x^2 - y^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta\,, \tag{98}$$

where θ is the angle which the line joining the origin to the point (x, y) makes with the positive x-direction. From (98) we conclude that

$$\epsilon = \theta \pm \pi \qquad \text{or} \qquad \epsilon = \theta \pm \frac{\pi}{2}\,. \tag{99}$$

Hence, *the principal axes of the velocity ellipse at any point are along the radial and the transverse directions, respectively.* In other words, we do not have any "deviation of the vertex."

The ratio of the axes of the velocity ellipse defined by (96) is also easily found. For, if $2l$ is the length of a principal axis, it should satisfy the equation

$$\frac{1}{l^4} - \frac{1}{l^2}(a+b) + (ab - h^2) = 0\,, \tag{100}$$

or

$$\frac{1}{l^2} = \tfrac{1}{2}\left\{(a+b) \pm \sqrt{(a+b)^2 - 4(ab-h^2)}\right\}. \tag{101}$$

Substituting for a, b, and h from (94) in (101), we find, after some simplifications, that

$$\frac{1}{l^2} = \tfrac{1}{2}\{2\kappa_1 + \kappa_2(x^2+y^2) \pm \kappa_2(x^2+y^2)\}\,. \tag{102}$$

If l_1 and l_2 denote the lengths of the semiaxes of the velocity ellipse, then, according to (102),

$$l_1^2 = \frac{1}{\kappa_1} ; \qquad l_2^2 = \frac{1}{\kappa_1 + \kappa_2(x^2 + y^2)} . \tag{103}$$

Hence, the ratio of the axes is given by

$$\frac{l_2}{l_1} = \sqrt{\frac{\kappa_1}{\kappa_1 + \kappa_2(x^2 + y^2)}} . \tag{104}$$

Considering next the motions (remembering that, according to our present choice of the origin of fundamental frame of reference, $\delta_1 = \delta_2 = 0$), we have

$$\left.\begin{aligned} aU_0 + hV_0 &= \Delta_1 = \beta y , \\ hU_0 + bV_0 &= \Delta_2 = -\beta x . \end{aligned}\right\} \tag{105}$$

Solving for U_0 and V_0, we obtain

$$\left.\begin{aligned} U_0 &= \frac{\beta}{ab - h^2} (hx + by) , \\ V_0 &= \frac{-\beta}{ab - h^2} (hy + ax) . \end{aligned}\right\} \tag{106}$$

From equation (94) we find that

$$\left.\begin{aligned} hx + by = \kappa_1 y ; \qquad ax + hy = \kappa_1 x , \\ ab - h^2 = \kappa_1[\kappa_1 + \kappa_2(x^2 + y^2)] . \end{aligned}\right\} \tag{107}$$

From (106) and (107) we now obtain

$$U_0 = \frac{\beta y}{\kappa_1 + \kappa_2(x^2 + y^2)} ; \qquad V_0 = -\frac{\beta x}{\kappa_1 + \kappa_2(x^2 + y^2)} . \tag{108}$$

If we denote by Θ_0 and Π_0 the rotational and the dilatational velocities, respectively, then

$$\Theta_0 = V_0 \cos\theta - U_0 \sin\theta = \frac{1}{\sqrt{x^2 + y^2}} (xV_0 - yU_0) , \tag{109}$$

$$\Pi_0 = V_0 \sin\theta + U_0 \cos\theta = \frac{1}{\sqrt{x^2 + y^2}} (yV_0 + xU_0) . \tag{110}$$

From (108), (109), and (110) it now follows that

$$\Theta_0 = -\frac{\beta(x^2+y^2)^{1/2}}{\kappa_1+\kappa_2(x^2+y^2)}\,; \qquad \Pi_0 = 0\,. \tag{111}$$

Hence, *the motions of the local centroids are of a purely rotational kind with no dilatational motions.*

If we denote the radial distance from the origin by $\bar{\omega}$, then (104) and (111) take the forms

$$\frac{l_2}{l_1} = \sqrt{\frac{\kappa_1}{\kappa_1+\kappa_2\bar{\omega}^2}}\,; \qquad \Theta_0 = -\frac{\beta\bar{\omega}}{\kappa_1+\kappa_2\bar{\omega}^2}\,. \tag{112}$$

If ω_0 denotes the angular velocity, then

$$\omega_0 = \frac{\Theta_0}{\bar{\omega}} = -\frac{\beta}{\kappa_1+\kappa_2\bar{\omega}^2}\,. \tag{113}$$

Hence, the Oort constant A is given by[9]

$$A = \frac{\bar{\omega}}{2}\frac{\partial\omega_0}{\partial\bar{\omega}} = \frac{\kappa_2\beta\bar{\omega}^2}{(\kappa_1+\kappa_2\bar{\omega}^2)^2} \tag{114}$$

or

$$\frac{A}{\omega_0} = -\frac{\kappa_2\bar{\omega}^2}{\kappa_1+\kappa_2\bar{\omega}^2}\,. \tag{115}$$

Hence, finally,

$$1+\frac{A}{\omega_0} = \frac{\kappa_1}{\kappa_1+\kappa_2\bar{\omega}^2} = \left(\frac{l_2}{l_1}\right)^2, \tag{116}$$

which is a relation first derived by Oort and Lindblad.

We have thus shown that, if the compatibility conditions are to be satisfied identically, then we are necessarily led to the Oort-Lindblad systems having (1) circular symmetry, (2) no differential dilatational motions, (3) no deviation of the vertex, and (4) a relation between the ratio of the axes of the velocity ellipse and the Oort constant A. We should, however, notice that we have proved (1) the circular symmetry and (2) the absence of differential dilatational

[9] Cf. W. M. Smart, *Stellar Dynamics*, pp. 393–395, Cambridge, Eng., 1938.

motions; we did not assume these characteristics, as is done in the methods of Oort and Lindblad.

Case 2: $S \equiv S(\tau)$.—If the compatibility conditions are not satisfied identically, then, as we have already seen, we should require of S (as defined in Eq. [85]) that it is a function of τ only. This is equivalent to restricting S to satisfy the condition (Eq. [86])

$$y\frac{\partial S}{\partial x} = x\frac{\partial S}{\partial y}. \tag{117}$$

On substituting for S from equation (85) and using our general solutions for a, b, and h (Eqs. [59], [60], and [57]), we find that (117) can be reduced to

$$\left.\begin{aligned} y^3\{h_2h_1 + h_3(b_0 - a_0)\} + h_2h_1x^2y + 2h_2^2x^3y - 2h_1h_3xy^2 \\ - 4h_2h_3x^2y^2 + 2h_3^2xy^3 = x^3\{h_1h_3 + h_2(a_0 - b_0)\} + h_3h_1y^2x \\ + 2h_3^2y^3x - 2h_1h_2x^2y - 4h_2h_3x^2y^2 + 2h_2^2yx^3\,. \end{aligned}\right\} \tag{118}$$

The terms x^3y, x^2y^2, and xy^3 cancel. Equating the coefficients of y^3, x^3, x^2y, and xy^2 to zero, we obtain

$$\left.\begin{aligned} h_1h_2 + h_3(b_0 - a_0) = 0; \qquad h_1h_3 = 0\,, \\ h_1h_3 + h_2(a_0 - b_0) = 0; \qquad h_1h_2 = 0\,. \end{aligned}\right\} \tag{119}$$

The equations (119) can be satisfied in either of the following two ways:

$$h_1 = 0, \qquad a_0 = b_0, \qquad h_2,\ h_3 \text{ arbitrary} \qquad (\text{case } \mathbf{a}_1) \tag{120}$$

or

$$h_1 \neq 0, \qquad h_2 = h_3 = 0, \qquad a_0,\ b_0 \text{ arbitrary} \qquad (\text{case } \mathbf{a}_2)\,. \tag{121}$$

Consider first the case $\mathbf{a}_1$. Then

$$\left.\begin{aligned} a &= -2h_2y - h_4y^2 - a_0\,, \\ b &= -2h_3x - h_4x^2 - a_0\,, \\ h &= h_2x + h_3y + h_4xy\,. \end{aligned}\right\} \tag{122}$$

The solutions for the coefficients of the velocity ellipse now involve four arbitrary constants. On substituting the foregoing expressions

for a, b, and h in the equation defining S, we find that it now reduces to

$$S = -\frac{3}{x^2 + y^2} = -\frac{3}{2\tau} . \tag{123}$$

On the other hand, from equation (84) we now infer that

$$\frac{\dfrac{d^2\mathfrak{B}}{d\tau^2}}{\dfrac{d\mathfrak{B}}{d\tau}} = -\frac{3}{2\tau} , \tag{124}$$

or, integrating,

$$\frac{d\mathfrak{B}}{d\tau} = \frac{\text{constant}}{\tau^{3/2}} . \tag{125}$$

Since, however, $\bar{\omega}^2 = 2\tau$ (cf. Eq. [80]), we easily find that (125) is equivalent to

$$\frac{d\mathfrak{B}}{d\bar{\omega}} = \frac{\text{constant}}{\bar{\omega}^2} , \tag{126}$$

which means that in *this case we have an inverse square law of force.*

Consider next the case α_2. Then

$$\left.\begin{aligned} a &= -h_4 y^2 - a_0 , \\ b &= -h_4 x^2 - b_0 , \\ h &= h_1 + h_4 xy . \end{aligned}\right\} \tag{127}$$

Further, from equation (85) it immediately follows that $S = 0$, or, according to (84),

$$\frac{d^2\mathfrak{B}}{d\tau^2} = 0 . \tag{128}$$

Hence,

$$\frac{d\mathfrak{B}}{d\tau} = \text{constant} , \tag{129}$$

or

$$\frac{d\mathfrak{B}}{d\bar{\omega}} = \text{constant } \bar{\omega} , \tag{130}$$

which is the case of a quasi-elastic field of force.

For the cases α_1 and α_2 just considered, we have deviations of the vertex predicted, and, further, the motions of the local centroids are no longer (for these cases) of a purely rotational kind.

We now see that the discussion of the integrability condition (in the form Eq. [78]) has shown that, while the general solutions of the differential equations (I_2) for the coefficients of the velocity ellipse contain six arbitrary constants, the integrability condition reduces this number. Thus, if equation (78) is to be satisfied identically (i.e., with no restriction on the form of $\mathfrak{V}(\bar{\omega})$), then there are only two arbitrary constants at our disposal. On the other hand, the number of constants at our disposal increases to four for certain special forms of $\mathfrak{V}(\bar{\omega})$, namely, for the cases of inverse square law and quasi-elastic fields of forces.

The results proved in this section have been derived by different methods by Lindblad[10] and Heckmann[11] on the basis of an earlier investigation by Shiveshwarkar.[12] But our analysis is more general than that of these authors. In particular, we have based our discussion on a more general form for the distribution function than was assumed by Shiveshwarkar. Further, while the authors quoted assume circular symmetry, we have proved this as a requirement for the existence of differential motions in two dimensions.

10. *The discussion of the compatibility conditions (case $\beta = 0$).*—For the case $\beta = 0$ we have already shown in § 8 that $\mathfrak{V}$ is constant along lines parallel to $\delta_2 x - \delta_1 y = 0$. By a rotation of the axes of the fundamental frame of reference we can arrange the equipotential lines to be parallel to the x-axis. We shall assume that this has been done. Then

$$\mathfrak{V}(x, y) \equiv \mathfrak{V}(y) . \tag{131}$$

For this choice of the orientation of the co-ordinate axes we have

$$\Delta_1 = \delta_1 ; \qquad \Delta_2 = 0 . \tag{132}$$

[10] *M.N.*, **96**, 69, 1935.

[11] *M.N.*, **96**, 67, 1935.

[12] *M.N.*, **95**, 655, 1935. For an account of these investigations see Smart, *op. cit.*, pp. 414–24.

Consider the integrability condition in the form (79). According to equation (79), we now have (since $\partial \mathfrak{V}/\partial x = 0$)

$$\left(\frac{\partial h}{\partial y} - \frac{\partial b}{\partial x}\right)\frac{d\mathfrak{V}}{dy} + h\,\frac{d^2\mathfrak{V}}{dy^2} = 0\ ; \tag{133}$$

or, substituting for a, b, and h according to equations (59), (60), and (57), we have

$$\frac{\dfrac{d^2\mathfrak{V}}{dy^2}}{\dfrac{d\mathfrak{V}}{dy}} = -3\,\frac{h_3 + h_4 x}{h_1 + h_2 x + h_3 y + h_4 xy}\,. \tag{134}$$

Hence, the integrability condition (133) will be satisfied identically if the right side of (134) is of the form 0/0 or

$$h_1 = h_2 = h_3 = h_4 = 0\,, \tag{135}$$

in which case

$$a = -a_0\,, \qquad b = -b_0\,, \qquad h = 0\,. \tag{136}$$

On the other hand, if the integrability condition is not satisfied identically, then the right-hand side of (134) should be independent of x. This is possible in either of the two following cases:

$$h_3 = h_4 = 0\,, \qquad h_1,\ h_2 \text{ arbitrary} \qquad (\text{case } \alpha_1) \tag{137}$$

or

$$h_1 = h_2 = 0\,, \qquad h_3,\ h_4 \text{ arbitrary} \qquad (\text{case } \alpha_2)\,. \tag{138}$$

In case α_1

$$\left.\begin{aligned} a &= -2h_2 y - a_0\,, \\ b &= -b_0\,, \\ h &= h_1 + h_2 x\,. \end{aligned}\right\} \tag{139}$$

Further, from (134) we have for this case

$$\frac{d^2\mathfrak{V}}{dy^2} = 0 \qquad \text{or} \qquad \frac{d\mathfrak{V}}{dy} = \text{constant}\,. \tag{140}$$

Hence, for the case α_1, $\mathfrak{B}$ is a constant (since it is independent of both x and y).

In case α_2

$$\left.\begin{aligned} a &= -h_4 y^2 - a_0 , \\ b &= -2h_3 x - h_4 x^2 - b_0 , \\ h &= h_3 y + h_4 xy . \end{aligned}\right\} \tag{141}$$

Further, from (134) we now have

$$\frac{\dfrac{d^2\mathfrak{B}}{dy^2}}{\dfrac{d\mathfrak{B}}{dy}} = -\frac{3}{y} , \tag{142}$$

or

$$\frac{d\mathfrak{B}}{dy} = \frac{\text{constant}}{y^3} . \tag{143}$$

11. *Stellar systems in two dimensions with differential motions and with no restriction on the form of* $\mathfrak{B}$.—In § 8 we saw that the case $\beta = \delta_1 = \delta_2 = 0$ is a "singular" one. It implies that

$$\left.\begin{aligned} aU_0 + hV_0 &= 0 , \\ hU_0 + bV_0 &= 0 . \end{aligned}\right\} \tag{144}$$

In this singular case there is no restriction on the form of $\mathfrak{B}$. However, (144) now implies that either $U_0 = V_0 = 0$ or $ab - h^2 = 0$. In the former case we have no differential motions, and, consequently, this case is not of much interest. If $U_0, V_0 \neq 0$, then

$$ab - h^2 = 0 , \tag{145}$$

or, using our general solution for a, b, and h, obtained in § 6,

$$(h_1 + h_2 x + h_3 y + h_4 xy)^2 = (a_0 + 2h_2 y + h_4 y^2)(b_0 + 2h_3 x + h_4 x^2) . \tag{146}$$

Expanding (146) and equating the coefficients of the different power combinations of x and y, we obtain

$$\left.\begin{aligned} h_1^2 &= a_0 b_0 \,; \qquad & h_1 h_4 &= h_2 h_3 \,, \\ h_2^2 &= a_0 h_4 \,; \qquad & h_1 h_2 &= h_3 a_0 \,, \\ h_3^2 &= b_0 h_4 \,; \qquad & h_1 h_3 &= h_2 b_0 \,. \end{aligned}\right\} \tag{147}$$

The six relations (147) are seen to be equivalent to

$$\left.\begin{aligned} a_0 &= \frac{h_2^2}{h_4} = \frac{h_1 h_2}{h_3} \,, \\ b_0 &= \frac{h_3^2}{h_4} = \frac{h_1 h_3}{h_2} \,. \end{aligned}\right\} \tag{148}$$

Hence,

$$a = -2h_2 y - h_4 y^2 - \frac{h_2^2}{h_4} = -h_4\left(y + \frac{h_2}{h_4}\right)^2 , \tag{149}$$

$$b = -2h_3 x - h_4 x^2 - \frac{h_3^2}{h_4} = -h_4\left(x + \frac{h_3}{h_4}\right)^2 . \tag{150}$$

Since $h^2 = ab$, we have

$$h = h_4\left(x + \frac{h_3}{h_4}\right)\left(y + \frac{h_2}{h_4}\right) . \tag{151}$$

Let

$$X = x + \frac{h_3}{h_4} \,; \qquad Y = y + \frac{h_2}{h_4} \,. \tag{152}$$

This corresponds to a parallel translation of the origin of our system of co-ordinates to $(-h_3/h_4, -h_2/h_4)$. Equations (149), (150), and (151) now take the simpler forms

$$a = \kappa Y^2 \,, \qquad b = \kappa X^2 \,, \qquad h = -\kappa XY \,, \tag{153}$$

where we have written κ in place of $-h_4$. Further, κ is an arbitrary constant. From (144) we now find that

$$\frac{U_0}{V_0} = -\frac{h}{a} = -\frac{b}{h} = \frac{X}{Y} \,. \tag{154}$$

Hence, in the system of co-ordinates (X, Y) *the motions of the local centroids are purely radial.*

For the case under consideration the fundamental quadratic form Q takes a remarkably simple form. Since

$$Q = a(U - U_0)^2 + b(V - V_0)^2 + 2h(U - U_0)(V - V_0) , \quad (155)$$

we have, according to (153),

$$Q = \kappa\{(U - U_0)Y - (V - V_0)X\}^2 , \quad (156)$$

or, using (154),

$$Q = \kappa(UY - VX)^2 . \quad (157)$$

Considering next the compatibility conditions (IV_3), we can now write them as

$$\left.\begin{aligned} a\frac{\partial \mathfrak{B}}{\partial X} + h\frac{\partial \mathfrak{B}}{\partial Y} &= -\frac{1}{2}\frac{\partial \chi}{\partial X} , \\ h\frac{\partial \mathfrak{B}}{\partial X} + b\frac{\partial \mathfrak{B}}{\partial Y} &= -\frac{1}{2}\frac{\partial \chi}{\partial Y} , \end{aligned}\right\} \quad (158)$$

where, according to (55),

$$\chi = -\sigma \quad (159)$$

since now $Q_0 = 0$ (cf. Eqs. [154] and [157]). From (153) and (158) we obtain

$$-\frac{1}{2}\frac{\partial \chi}{\partial X} = \kappa Y\left(Y\frac{\partial \mathfrak{B}}{\partial X} - X\frac{\partial \mathfrak{B}}{\partial Y}\right) , \quad (160)$$

$$\frac{1}{2}\frac{\partial \chi}{\partial Y} = \kappa X\left(Y\frac{\partial \mathfrak{B}}{\partial X} - X\frac{\partial \mathfrak{B}}{\partial Y}\right) , \quad (161)$$

or

$$X\frac{\partial \chi}{\partial X} + Y\frac{\partial \chi}{\partial Y} = 0 . \quad (162)$$

Hence,

$$\chi \equiv \chi\left(\frac{X}{Y}\right) . \quad (163)$$

Finally, for the distribution function we have

$$\Psi = \Psi[\kappa(UY - VX)^2 + \sigma] , \quad (164)$$

where, according to (159) and (163),

$$\sigma \equiv \sigma\left(\frac{X}{Y}\right). \tag{165}$$

If we choose a Gaussian form for the distribution function, we have

$$\Psi = \rho\left(\frac{X}{Y}\right)e^{-\kappa(UY-VX)^2}, \tag{166}$$

where $\rho = e^{-\sigma}$; in other words, the equi-density lines are radial and we have a Maxwellian distribution of the angular momenta.

III. THE SOLUTIONS FOR THE COEFFICIENTS OF THE VELOCITY ELLIPSOID AND THE MOTIONS IN CARTESIAN CO-ORDINATES

12. *The fundamental equations in Cartesian co-ordinates.*—For the fundamental frame of reference we shall choose a Cartesian system of co-ordinates (x, y, z). For such a system $P = Q = R = 1$, and the differential equations (I), (II), (III), and (IV) now become

$$\left.\begin{array}{ll}
\dfrac{\partial a}{\partial x} = 0, \quad \text{(i)}; \qquad \dfrac{\partial b}{\partial y} = 0, \quad \text{(ii)}; & \dfrac{\partial c}{\partial z} = 0, \quad \text{(iii)} \\[2ex]
2\dfrac{\partial h}{\partial x} + \dfrac{\partial a}{\partial y} = 0, \quad \text{(iv)}; & 2\dfrac{\partial g}{\partial x} + \dfrac{\partial a}{\partial z} = 0, \quad \text{(v)} \\[2ex]
2\dfrac{\partial h}{\partial y} + \dfrac{\partial b}{\partial x} = 0, \quad \text{(vi)}; & 2\dfrac{\partial f}{\partial y} + \dfrac{\partial b}{\partial z} = 0, \quad \text{(vii)} \\[2ex]
2\dfrac{\partial g}{\partial z} + \dfrac{\partial c}{\partial x} = 0, \quad \text{(viii)}; & 2\dfrac{\partial f}{\partial z} + \dfrac{\partial c}{\partial y} = 0, \quad \text{(ix)} \\[2ex]
\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z} = 0, & \text{(x)}
\end{array}\right\} \tag{I$_4$}$$

$$\left.\begin{array}{ll}
\dfrac{\partial \Delta_1}{\partial x} = 0, \quad \text{(i)}; & \dfrac{\partial \Delta_2}{\partial x} + \dfrac{\partial \Delta_1}{\partial y} = 0, \quad \text{(iv)} \\[2ex]
\dfrac{\partial \Delta_2}{\partial y} = 0, \quad \text{(ii)}; & \dfrac{\partial \Delta_3}{\partial y} + \dfrac{\partial \Delta_2}{\partial z} = 0, \quad \text{(v)} \\[2ex]
\dfrac{\partial \Delta_3}{\partial z} = 0, \quad \text{(iii)}; & \dfrac{\partial \Delta_1}{\partial z} + \dfrac{\partial \Delta_3}{\partial x} = 0, \quad \text{(vi)}
\end{array}\right\} \tag{II$_4$}$$

$$\Delta_1 \frac{\partial \mathfrak{B}}{\partial x} + \Delta_2 \frac{\partial \mathfrak{B}}{\partial y} + \Delta_3 \frac{\partial \mathfrak{B}}{\partial z} = 0, \tag{III$_4$}$$

and

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \operatorname{grad} \mathfrak{V} = -\tfrac{1}{2} \operatorname{grad} \chi \,. \qquad (\mathrm{IV}_4)$$

13. *The solution for the coefficients of the velocity ellipsoid.*—A simple examination of the differential equations (I_4) reveals that

$$\left.\begin{array}{l} a \text{ is independent of } x \text{ and quadratic in } y \text{ and } z \,, \\ b \text{ is independent of } y \text{ and quadratic in } z \text{ and } x \,, \\ c \text{ is independent of } z \text{ and quadratic in } x \text{ and } y \,, \\ f \text{ is linear in } y \text{ and } z \text{ and quadratic in } x \,, \\ g \text{ is linear in } z \text{ and } x \text{ and quadratic in } y \,, \\ h \text{ is linear in } x \text{ and } y \text{ and quadratic in } z \,. \end{array}\right\} \qquad (167)$$

The solutions for f, g, and h must therefore be of the following forms:

$$f = f_1 + f_2 y + f_3 z + f_4 yz \,, \qquad (168)$$

$$g = g_1 + g_2 z + g_3 x + g_4 zx \,, \qquad (169)$$

$$h = h_1 + h_2 x + h_3 y + h_4 xy \,, \qquad (170)$$

where $(f_1, \ldots, f_4)$, $(g_1, \ldots, g_4)$, and $(h_1, \ldots, h_4)$ are general polynomials of the second degree in x, y, and z, respectively. We can therefore write

$$f_n = f_{n0} + f_{n1} x + f_{n2} x^2 \qquad (n = 1, \ldots, 4) \,, \qquad (171)$$

$$g_n = g_{n0} + g_{n1} y + g_{n2} y^2 \qquad (n = 1, \ldots, 4) \,, \qquad (172)$$

$$h_n = h_{n0} + h_{n1} z + h_{n2} z^2 \qquad (n = 1, \ldots, 4) \,, \qquad (173)$$

where $f_{n0}, \ldots, h_{n2}$ are all constants, arbitrary in the first instance. Differentiating equations (iv) and (v) partially with respect to z and y, respectively, we find that

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 h}{\partial x \partial z} \,. \qquad (174)$$

Similarly from (vi) and (vii) and from (viii) and (ix), we find that

$$\frac{\partial^2 h}{\partial y \partial z} = \frac{\partial^2 f}{\partial y \partial x} \,; \qquad \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 g}{\partial z \partial y} \,. \qquad (175)$$

Substituting for f, g, and h from equations (168)–(170) in (174) and (175), we have

$$\left.\begin{aligned}\frac{\partial}{\partial y}(g_3+g_4z) &= \frac{\partial}{\partial z}(h_2+h_4y)\,,\\ \frac{\partial}{\partial z}(h_3+h_4x) &= \frac{\partial}{\partial x}(f_2+f_4z)\,,\\ \frac{\partial}{\partial x}(f_3+f_4y) &= \frac{\partial}{\partial y}(g_2+g_4x)\,.\end{aligned}\right\}\tag{176}$$

From equations (171), (172), (173), and (176) we now find

$$\left.\begin{aligned}g_{31}+2g_{32}y+z(g_{41}+2g_{42}y) &= h_{21}+2h_{22}z+y(h_{41}+2h_{42}z)\,,\\ h_{31}+2h_{32}z+x(h_{41}+2h_{42}z) &= f_{21}+2f_{22}x+z(f_{41}+2f_{42}x)\,,\\ f_{31}+2f_{32}x+y(f_{41}+2f_{42}x) &= g_{21}+2g_{22}y+x(g_{41}+2g_{42}y)\,.\end{aligned}\right\}\tag{177}$$

Equating the coefficients of x, y, z, etc., in the equations (177), we find

$$\left.\begin{aligned}&g_{31}=h_{21}\,; && g_{41}=2h_{22}\,; && 2g_{32}=h_{41}\,; && g_{42}=h_{42}\,,\\ &h_{31}=f_{21}\,; && h_{41}=2f_{22}\,; && 2h_{32}=f_{41}\,; && h_{42}=f_{42}\,,\\ &f_{31}=g_{21}\,; && f_{41}=2g_{22}\,; && 2f_{32}=g_{41}\,; && f_{42}=g_{42}\,,\end{aligned}\right\}\tag{178}$$

or, after some simplifications,

$$\left.\begin{aligned}&f_{21}=h_{31}\,; && f_{41}=2g_{22}=2h_{32}\,,\\ &g_{21}=f_{31}\,; && g_{41}=2h_{22}=2f_{32}\,,\\ &h_{21}=g_{31}\,; && h_{41}=2f_{22}=2g_{32}\,,\\ &f_{42}=g_{42}=h_{42}\,.\end{aligned}\right\}\tag{179}$$

On substituting equations (168)–(173) in equation (x) of (I_4), we obtain

$$\left.\begin{aligned}&(f_{11}+2f_{12}x)+(f_{21}+2f_{22}x)y+(f_{31}+2f_{32}x)z+(f_{41}+2f_{42}x)yz\\ &\quad+(g_{11}+2g_{12}y)+(g_{21}+2g_{22}y)z+(g_{31}+2g_{32}y)x\\ &\quad+(g_{41}+2g_{42}y)zx+(h_{11}+2h_{12}z)+(h_{21}+2h_{22}z)x\\ &\quad+(h_{31}+2h_{32}z)y+(h_{41}+2h_{42}z)xy=0\,.\end{aligned}\right\}\tag{180}$$

On equating the coefficients of x, y, etc., we find

$$\left.\begin{aligned} f_{11} + g_{11} + h_{11} &= 0\,, \\ 2f_{12} + g_{31} + h_{21} &= 0\,, \\ f_{21} + 2g_{12} + h_{31} &= 0\,, \\ f_{31} + g_{21} + 2h_{12} &= 0\,, \\ 2f_{22} + 2g_{32} + h_{41} &= 0\,, \\ f_{41} + 2g_{22} + 2h_{32} &= 0\,, \\ 2f_{32} + g_{41} + 2h_{22} &= 0\,, \\ f_{42} + g_{42} + h_{42} &= 0\,. \end{aligned}\right\} \tag{181}$$

From equations (179) and (181) we easily find that

$$\left.\begin{aligned} f_{12} &= -h_{21} = -g_{31}\,; & f_{22} &= g_{32} = h_{41} = h_{42} = 0\,, \\ g_{12} &= -f_{21} = -h_{31}\,; & g_{22} &= h_{32} = f_{41} = f_{42} = 0\,, \\ h_{12} &= -g_{21} = -f_{31}\,; & h_{22} &= f_{32} = g_{41} = g_{42} = 0\,. \end{aligned}\right\} \tag{182}$$

Hence, our solutions for f, g, and h are

$$f = (f_{10} + f_{11}x - h_{21}x^2) + (f_{20} + f_{21}x)y + (f_{30} + g_{21}x)z + f_{40}yz\,, \tag{183}$$

$$g = (g_{10} + g_{11}y - f_{21}y^2) + (g_{20} + g_{21}y)z + (g_{30} + h_{21}y)x + g_{40}zx\,, \tag{184}$$

$$h = (h_{10} + h_{11}z - g_{21}z^2) + (h_{20} + h_{21}z)x + (h_{30} + f_{21}z)y + h_{40}xy\,. \tag{185}$$

In the foregoing expressions for f, g, and h the coefficients f_{10}, g_{10}, h_{10}, f_{20}, g_{20}, h_{20}, f_{21}, g_{21}, h_{21}, f_{30}, g_{30}, h_{30}, f_{40}, g_{40}, and h_{40} are all arbitrary constants. Of the remaining three coefficients, f_{11}, g_{11}, and h_{11}, two are arbitrary and the third has to be found from the relation (cf. Eq. [181])

$$f_{11} + g_{11} + h_{11} = 0\,. \tag{186}$$

The solutions for a, b, and c now readily follow. Thus from equation (iv) of (I_4) we have

$$\frac{\partial a}{\partial y} = -2\,\frac{\partial h}{\partial x} = -2(h_{20} + h_{21}z) - 2h_{40}y\,. \tag{187}$$

On the other hand, according to (167), a is independent of x and quadratic in y and z. Hence, on integrating (187), we have

$$a = -2(h_{20} + h_{21}z)y - h_{40}y^2 - (a_0 + a_1 z + a_2 z^2) , \qquad (188)$$

where a_0, a_1, and a_2 are constants. From equations (v) of (I_4), (184), and (188) we now have

$$2h_{21}y + a_1 + 2a_2 z = 2g_{30} + 2h_{21}y + 2g_{40}z . \qquad (189)$$

Hence,

$$a_1 = 2g_{30}, \quad a_2 = g_{40} . \qquad (190)$$

Our solution for a is therefore given by

$$a = -2(h_{20} + h_{21}z)y - h_{40}y^2 - (a_0 + 2g_{30}z + g_{40}z^2) . \qquad (191)$$

Similarly,

$$b = -2(f_{20} + f_{21}x)z - f_{40}z^2 - (b_0 + 2h_{30}x + h_{40}x^2) , \qquad (192)$$

$$c = -2(g_{20} + g_{21}y)x - g_{40}x^2 - (c_0 + 2f_{30}y + f_{40}y^2) . \qquad (193)$$

Equations (183), (184), (185), (191), (192), and (193) represent, then, the general solution of the ten partial differential equations (I_4). Our general expressions for the coefficients of the velocity ellipsoid a, b, c, f, g, and h are seen to involve twenty arbitrary constants.

Now the equations (I) for a, b, c, f, g, and h do not involve, and are therefore independent of, the existence or nonexistence of differential motions. Hence, the solutions we have obtained are valid also for stellar systems with no differential motions.[13] When there are no differential motions, the dynamical problem formulated in § 2 now reduces to finding the circumstances under which the equation of continuity admits of a solution of the form

$$\left.\begin{aligned}\Psi(x, y, z) \equiv \Psi(aU^2 + bV^2 + cW^2 + 2fVW \\ + 2gWU + 2hUV + \sigma) ,\end{aligned}\right\} \qquad (194)$$

where a, b, c, f, g, h, and σ are continuous functions of position. This problem was first formulated by Eddington.[14] Eddington's solu-

[13] This would be the case if the different local frames of references are at relative rest.

[14] *M.N.*, **76**, 37, 1915.

tion[15] contains, however, only three arbitrary constants, as against the twenty that we have found. This discrepancy arises from the circumstance that Eddington has obtained only an extremely special solution. In his investigation Eddington introduces the notion of the "principal velocity surfaces," a notion which is seen to be fallacious.[16] It is true that we can regard the directions of the principal axes of the velocity ellipsoid at any given point as being tangential to three *curves* which intersect orthogonally at the point considered. But it is *not* generally true that we can regard these curves as the intersections of a triply orthogonal system of *surfaces*. Consequently, the notion of the principal velocity surfaces introduces restrictions of a mathematical character (cf. n. 16) on the physical problem which are wholly irrelevant. It should be further pointed out that the so-called "Eddington's theorem," that "the principal velocity surfaces form a set of confocal quadrics," has no validity as a general theorem in stellar dynamics. Finally, reference should also be made to an investigation by Jeans[17] where the dynamics of stellar systems

[15] *Ibid.*, p. 46.

[16] The fallacy becomes clear when an attempt is made to make the notion of the "principal velocity surfaces" explicit.

Let $l(x, y, z)$, $m(x, y, z)$, and $n(x, y, z)$ be the direction cosines of one of the principal axes of the velocity ellipsoid. Let us suppose that there exists a one-parametric family of surfaces, $\sigma(x, y, z) =$ constant, such that the vector (l, m, n) at the point (x, y, z) is normal to the surface, belonging to the family $\sigma =$ constant, which passes through the point under consideration. We should then have

$$l = \tau \frac{\partial \sigma}{\partial x}\,; \qquad m = \tau \frac{\partial \sigma}{\partial y}\,; \qquad n = \tau \frac{\partial \sigma}{\partial z}\,, \tag{i}$$

where τ is some function of the space co-ordinates. On the other hand, from the equation $\sigma =$ constant, we obtain

$$\frac{\partial \sigma}{\partial x} dx + \frac{\partial \sigma}{\partial y} dy + \frac{\partial \sigma}{\partial z} dz = 0\,. \tag{ii}$$

From (i) and (ii) we obtain

$$l dx + m dy + n dz = 0\,. \tag{iii}$$

Equations (i) and (ii) now imply that the Pfaffian differential equation (iii) admits of an integrating denominator. But it is well known that a Pfaffian differential equation in more than two variables will not, in general, admit of an integrating denominator. Indeed, one of the famous theorems (due to Carathéodory) in the theory of Pfaffian equations is concerned precisely with this question (see S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* [Chicago, 1939], pp. 19–24).

[17] *M.N.*, **76**, 70, 1915.

with no differential motions is considered. But, unfortunately, Jeans's solution[18] is again not the most general one; his solution contains only fifteen constants of integration, as against the twenty which characterize the general solution that we have found.

14. *The solution for the motions of the local centroids.*—We shall now consider the six equations (II_4) for the Δ's. For this set of equations we readily find that

$$\left.\begin{aligned} \frac{\partial^2\Delta_1}{\partial y^2} &= 0 \, ; & \frac{\partial^2\Delta_1}{\partial z^2} &= 0 \, , \\ \frac{\partial^2\Delta_2}{\partial z^2} &= 0 \, ; & \frac{\partial^2\Delta_2}{\partial x^2} &= 0 \, , \\ \frac{\partial^2\Delta_3}{\partial x^2} &= 0 \, ; & \frac{\partial^2\Delta_3}{\partial y^2} &= 0 \, . \end{aligned}\right\} \tag{195}$$

From equations (i), (ii), (iii), and (195) we conclude that

$$\left.\begin{aligned} &\Delta_1 \text{ is independent of } x \text{ and linear in } y \text{ and } z \, , \\ &\Delta_2 \text{ is independent of } y \text{ and linear in } z \text{ and } x \, , \\ &\Delta_3 \text{ is independent of } z \text{ and linear in } x \text{ and } y \, . \end{aligned}\right\} \tag{196}$$

We can therefore write

$$\left.\begin{aligned} \Delta_1 &= \alpha_1 yz + \beta_3 y + \gamma_2 z + \delta_1 \, , \\ \Delta_2 &= \alpha_2 zx + \beta_1 z + \gamma_3 x + \delta_2 \, , \\ \Delta_3 &= \alpha_3 xy + \beta_2 x + \gamma_1 y + \delta_3 \, , \end{aligned}\right\} \tag{197}$$

where $\alpha_1, \alpha_2, \ldots, \delta_3$ are all constants. Substituting (197) in equations (iv), (v), and (vi) of (II_4), we obtain

$$\left.\begin{aligned} \alpha_2 z + \gamma_3 + \alpha_1 z + \beta_3 &= 0 \, , \\ \alpha_3 x + \gamma_1 + \alpha_2 x + \beta_1 &= 0 \, , \\ \alpha_1 y + \gamma_2 + \alpha_3 y + \beta_2 &= 0 \, . \end{aligned}\right\} \tag{198}$$

Hence,

$$\alpha_2 = -\alpha_1 \, ; \qquad \alpha_3 = -\alpha_2 \, ; \qquad \alpha_1 = -\alpha_3 \tag{199}$$

[18] *Ibid.*, p. 73.

and

$$\gamma_3 = -\beta_3 \; ; \qquad \gamma_1 = -\beta_1 \; ; \qquad \gamma_2 = -\beta_2 \, . \tag{200}$$

From (199) we see that

$$\alpha_1 = \alpha_2 = \alpha_3 = 0 \, . \tag{201}$$

Hence, our solution is

$$\Delta_1 = \beta_3 y - \beta_2 z + \delta_1 \, , \tag{202}$$

$$\Delta_2 = \beta_1 z - \beta_3 x + \delta_2 \, , \tag{203}$$

$$\Delta_3 = \beta_2 x - \beta_1 y + \delta_3 \, . \tag{204}$$

We thus see that the solution involves six arbitrary constants.

The solution for the Δ's can be written as a single vector equation as

$$\boldsymbol{\Delta} = \boldsymbol{r} \times \boldsymbol{\beta} + \boldsymbol{\delta} \, , \tag{205}$$

where $\boldsymbol{\Delta}$, $\boldsymbol{r}$, $\boldsymbol{\beta}$, and $\boldsymbol{\delta}$ represent the vectors $(\Delta_1, \Delta_2, \Delta_3)$, (x, y, z), $(\beta_1, \beta_2, \beta_3)$, and $(\delta_1, \delta_2, \delta_3)$, respectively.

IV. THE HELICAL SYMMETRY OF STELLAR SYSTEMS WITH DIFFERENTIAL MOTIONS

15. *The solution of the partial linear differential equation for* $\mathfrak{B}$.—According to (III$_4$), we can write

$$\boldsymbol{\Delta} \cdot \operatorname{grad} \mathfrak{B} = 0 \, , \tag{206}$$

where, as in § 14, $\boldsymbol{\Delta}$ stands for the vector $(\Delta_1, \Delta_2, \Delta_3)$. Hence,

$$\mathfrak{B} \equiv \mathfrak{B}(I_1, I_2) \, , \tag{207}$$

where I_1 and I_2 are any two independent first integrals of the Lagrangian subsidiary equations

$$\frac{dx}{\Delta_1} = \frac{dy}{\Delta_2} = \frac{dz}{\Delta_3} \, . \tag{208}$$

Substituting our solutions for Δ_1, Δ_2, and Δ_3 according to equations (202)–(204), we have

$$\frac{dx}{\beta_3 y - \beta_2 z + \delta_1} = \frac{dy}{\beta_1 z - \beta_3 x + \delta_2} = \frac{dz}{\beta_2 x - \beta_1 y + \delta_3} = \frac{dt}{t} \text{ (say)} . \quad (209)$$

One integral of (209) is immediately found. We have

$$\frac{dt}{t} = \frac{\beta_1 dx + \beta_2 dy + \beta_3 dz}{\boldsymbol{\beta} \cdot \boldsymbol{\Delta}} . \quad (210)$$

On the other hand, according to (205),

$$\boldsymbol{\beta} \cdot \boldsymbol{\Delta} = \boldsymbol{\beta} \cdot \boldsymbol{r} \times \boldsymbol{\beta} + \boldsymbol{\beta} \cdot \boldsymbol{\delta} = \boldsymbol{\beta} \cdot \boldsymbol{\delta} . \quad (210')$$

Hence,

$$\frac{dt}{t} = \frac{\beta_1 dx + \beta_2 dy + \beta_3 dz}{\boldsymbol{\beta} \cdot \boldsymbol{\delta}} , \quad (211)$$

or after integration

$$t = \gamma_3 e^{\boldsymbol{\beta} \cdot \boldsymbol{r} / \boldsymbol{\beta} \cdot \boldsymbol{\delta}} , \quad (212)$$

where γ_3 is a constant.

Let $\boldsymbol{\iota}_3 = (l_3, m_3, n_3)$ be a unit vector in the direction of $\boldsymbol{\beta}$. Then

$$l_3 = \frac{\beta_1}{|\boldsymbol{\beta}|} ; \qquad m_3 = \frac{\beta_2}{|\boldsymbol{\beta}|} ; \qquad n_3 = \frac{\beta_3}{|\boldsymbol{\beta}|} , \quad (213)$$

where $|\boldsymbol{\beta}|$ defines the length of the vector $\boldsymbol{\beta}$ according to

$$|\boldsymbol{\beta}| = \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} . \quad (214)$$

Further, let

$$\Sigma = \frac{|\boldsymbol{\beta}|^2}{\boldsymbol{\beta} \cdot \boldsymbol{\delta}} \boldsymbol{\iota}_3 \cdot \boldsymbol{r} . \quad (215)$$

We can now write (212) as

$$t = \gamma_3 e^{\Sigma / |\boldsymbol{\beta}|} . \quad (216)$$

To obtain two further integrals we proceed as follows:

From (209) we derive that

$$\frac{dt}{t}=\frac{ldx+mdy+ndz}{(\beta_2 n-\beta_3 m)x+(\beta_3 l-\beta_1 n)y+(\beta_1 m-\beta_2 l)z+l\delta_1+m\delta_2+n\delta_3}, \tag{217}$$

where l, m, and n can be arbitrary constants. Let l, m, and n be so chosen that

$$\left.\begin{aligned} -\beta_3 m+\beta_2 n&=\epsilon l\,,\\ \beta_3 l\qquad\quad-\beta_1 n&=\epsilon m\,,\\ -\beta_2 l+\beta_1 m\qquad\quad&=\epsilon n\,.\end{aligned}\right\} \tag{218}$$

Then

$$\frac{dt}{t}=\frac{ldx+mdy+ndz}{\epsilon(lx+my+nz+q)}, \tag{219}$$

where

$$\epsilon q=l\delta_1+m\delta_2+n\delta_3\,. \tag{220}$$

This choice of l, m, and n is possible only if ϵ is a root of the equation

$$\begin{vmatrix} -\epsilon & -\beta_3 & \beta_2\\ \beta_3 & -\epsilon & -\beta_1\\ -\beta_2 & \beta_1 & -\epsilon\end{vmatrix}=0\,, \tag{221}$$

i.e., ϵ is a characteristic root of the matrix

$$\begin{pmatrix} 0 & -\beta_3 & \beta_2\\ \beta_3 & 0 & -\beta_1\\ -\beta_2 & \beta_1 & 0\end{pmatrix}. \tag{222}$$

This matrix is of rank 2, and this implies that one of the eigenvalues of (222) is zero. In fact, on expanding the determinant (221), we have

$$\epsilon^3+\epsilon(\beta_1^2+\beta_2^2+\beta_3^2)=0\,. \tag{223}$$

Hence, the two nonzero eigenvalues ϵ_1 and ϵ_2 of (222) are

$$\epsilon_1=i|\boldsymbol{\beta}|\;;\qquad \epsilon_2=-i|\boldsymbol{\beta}|\,. \tag{224}$$

From (219) we have, on integration,

$$t = \gamma(lx + my + nz + q)^{1/\epsilon}, \tag{225}$$

where γ is a constant of integration. We have two integrals of the form (225), corresponding to the two eigenvalues ϵ_1 and ϵ_2 (cf. Eq. [224]). The equations (218) for l, m, and n can be written as

$$-\epsilon l - \beta_3 m + \beta_2 n = 0, \tag{226}$$

$$\beta_3 l - \epsilon m - \beta_1 n = 0, \tag{227}$$

$$-\beta_2 l + \beta_1 m - \epsilon n = 0. \tag{228}$$

Multiply the equations (226), (227), and (228) by β_1, β_2, and β_3, respectively, and add. We find

$$\beta_1 l + \beta_2 m + \beta_3 n = 0. \tag{229}$$

Again, multiply the equations (226), (227), and (228) by l, m, and n and add. We now obtain

$$l^2 + m^2 + n^2 = 0. \tag{230}$$

We have thus shown that the eigenvector, $\boldsymbol{\iota} = (l, m, n)$, of the matrix (222) is a complex vector of zero length and orthogonal to $\boldsymbol{\beta}$, and hence orthogonal also to $\boldsymbol{\iota}_3$. Let $\boldsymbol{\iota}_1$ and $\boldsymbol{\iota}_2$ be two unit (real) vectors orthogonal to each other and to $\boldsymbol{\iota}_3$. Then

$$\boldsymbol{\iota} = \boldsymbol{\iota}_2 \pm i\boldsymbol{\iota}_1 \tag{231}$$

is a complex vector readily verified to have the required characteristics. Thus, if

$$\boldsymbol{\iota}_1 = (l_1, m_1, n_1)\,; \qquad \boldsymbol{\iota}_2 = (l_2, m_2, n_2), \tag{232}$$

then

$$\left.\begin{aligned} l &= l_2 \pm il_1, \\ m &= m_2 \pm im_1, \\ n &= n_2 \pm in_1 \end{aligned}\right\} \tag{233}$$

satisfy the equations (226), (227), and (228), the plus sign in (233) corresponding to the eigenvalue $\epsilon_1 = i|\boldsymbol{\beta}|$ and the minus sign to $\epsilon_2 = -i|\boldsymbol{\beta}|$. Defined in this manner, $(\boldsymbol{\iota}_1, \boldsymbol{\iota}_2, \boldsymbol{\iota}_3)$ are readily seen to form a right-handed system of mutually orthogonal vectors. Hence,

$$\boldsymbol{\iota}_1 = \boldsymbol{\iota}_2 \times \boldsymbol{\iota}_3 ; \qquad \boldsymbol{\iota}_2 = \boldsymbol{\iota}_3 \times \boldsymbol{\iota}_1 . \tag{234}$$

Now, according to (220),

$$\pm i|\boldsymbol{\beta}|q = \boldsymbol{\iota} \cdot \boldsymbol{\delta} , \tag{235}$$

or, according to (231),

$$\pm i|\boldsymbol{\beta}|q = \boldsymbol{\iota}_2 \cdot \boldsymbol{\delta} \pm i\boldsymbol{\iota}_1 \cdot \boldsymbol{\delta} . \tag{236}$$

Hence,

$$q = \frac{1}{|\boldsymbol{\beta}|} (\boldsymbol{\iota}_1 \cdot \boldsymbol{\delta} \mp i\boldsymbol{\iota}_2 \cdot \boldsymbol{\delta}) . \tag{237}$$

From (224) and (225) we now have

$$t = \gamma\left[(\boldsymbol{\iota}_2 \pm i\boldsymbol{\iota}_1) \cdot \boldsymbol{r} + \frac{1}{|\boldsymbol{\beta}|} (\boldsymbol{\iota}_1 \mp i\boldsymbol{\iota}_2) \cdot \boldsymbol{\delta} \right]^{\mp i/|\boldsymbol{\beta}|} , \tag{238}$$

or, collecting the real and the imaginary terms,

$$t = \gamma\left[\left(\boldsymbol{\iota}_2 \cdot \boldsymbol{r} + \frac{1}{|\boldsymbol{\beta}|} \boldsymbol{\iota}_1 \cdot \boldsymbol{\delta} \right) \pm i\left(\boldsymbol{\iota}_1 \cdot \boldsymbol{r} - \frac{1}{|\boldsymbol{\beta}|} \boldsymbol{\iota}_2 \cdot \boldsymbol{\delta} \right) \right]^{\mp i/|\boldsymbol{\beta}|} . \tag{239}$$

Let

$$\mathfrak{A} = \boldsymbol{\iota}_2 \cdot \boldsymbol{r} + \frac{1}{|\boldsymbol{\beta}|} \boldsymbol{\iota}_1 \cdot \boldsymbol{\delta} \tag{240}$$

and

$$\mathfrak{B} = \boldsymbol{\iota}_1 \cdot \boldsymbol{r} - \frac{1}{|\boldsymbol{\beta}|} \boldsymbol{\iota}_2 \cdot \boldsymbol{\delta} . \tag{241}$$

Equation (239) now takes the form

$$t = \gamma(\mathfrak{A} \pm i\mathfrak{B})^{\mp i/|\boldsymbol{\beta}|} . \tag{242}$$

Combining equations (216) and (242), we finally have

$$\gamma_1(\mathfrak{A} + i\mathfrak{B})^{-i/|\boldsymbol{\beta}|} = \gamma_2(\mathfrak{A} - i\mathfrak{B})^{+i/|\boldsymbol{\beta}|} = \gamma_3 e^{\Sigma/|\boldsymbol{\beta}|} , \tag{243}$$

where, among the three constants, γ_1, γ_2, and γ_3, two are arbitrary. The equations (243) can be written alternatively in the forms

$$(\mathfrak{A} + i\mathfrak{B})e^{-i\Sigma} = (\kappa_1 + i\kappa_2) , \tag{244}$$

$$(\mathfrak{A} - i\mathfrak{B})e^{+i\Sigma} = (\kappa_1 - i\kappa_2) , \tag{245}$$

where κ_1 and κ_2 are two arbitrary (real) constants. Equations (244) and (245) are easily seen to be equivalent to

$$\left.\begin{aligned}(\mathfrak{A}\cos\Sigma + \mathfrak{B}\sin\Sigma) + i(\mathfrak{B}\cos\Sigma - \mathfrak{A}\sin\Sigma) &= (\kappa_1 + i\kappa_2) ,\\ (\mathfrak{A}\cos\Sigma + \mathfrak{B}\sin\Sigma) - i(\mathfrak{B}\cos\Sigma - \mathfrak{A}\sin\Sigma) &= (\kappa_1 - i\kappa_2) .\end{aligned}\right\} \tag{246}$$

From (246) we obtain

$$\left.\begin{aligned}\mathfrak{A}\cos\Sigma + \mathfrak{B}\sin\Sigma &= \kappa_1 ,\\ \mathfrak{B}\cos\Sigma - \mathfrak{A}\sin\Sigma &= \kappa_2 .\end{aligned}\right\} \tag{247}$$

Let

$$\tan\theta = \frac{\mathfrak{A}}{\mathfrak{B}} , \tag{248}$$

so that

$$\sin\theta = \frac{\mathfrak{A}}{\sqrt{\mathfrak{A}^2 + \mathfrak{B}^2}} ; \qquad \cos\theta = \frac{\mathfrak{B}}{\sqrt{\mathfrak{A}^2 + \mathfrak{B}^2}} . \tag{249}$$

Equations (247) can now be expressed in the forms

$$\sqrt{\mathfrak{A}^2 + \mathfrak{B}^2}\,\sin(\Sigma + \theta) = \kappa_1 , \tag{250}$$

$$\sqrt{\mathfrak{A}^2 + \mathfrak{B}^2}\,\cos(\Sigma + \theta) = \kappa_2 . \tag{251}$$

From (250) and (251) we finally obtain our two integrals

$$\mathfrak{A}^2 + \mathfrak{B}^2 = \text{constant} = I_1 , \tag{252}$$

$$\Sigma + \theta = \text{constant} = I_2 . \tag{253}$$

These two integrals have simple geometrical interpretations.

Now, according to (234) and (240),

$$\mathfrak{A} = \boldsymbol{\iota}_2 \cdot \boldsymbol{r} + \frac{1}{|\boldsymbol{\beta}|} (\boldsymbol{\iota}_2 \times \boldsymbol{\iota}_3) \cdot \boldsymbol{\delta} \tag{254}$$

$$= \boldsymbol{\iota}_2 \cdot \left(\boldsymbol{r} + \frac{\boldsymbol{\iota}_3 \times \boldsymbol{\delta}}{|\boldsymbol{\beta}|} \right) ; \tag{255}$$

or, since

$$\boldsymbol{\iota}_3 = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|} , \tag{256}$$

we have

$$\mathfrak{A} = \boldsymbol{\iota}_2 \cdot \left(\boldsymbol{r} + \frac{\boldsymbol{\beta} \times \boldsymbol{\delta}}{|\boldsymbol{\beta}|^2} \right) . \tag{257}$$

Again, according to (234) and (241),

$$\mathfrak{B} = \boldsymbol{\iota}_1 \cdot \boldsymbol{r} - \frac{1}{|\boldsymbol{\beta}|} (\boldsymbol{\iota}_3 \times \boldsymbol{\iota}_1) \cdot \boldsymbol{\delta} \tag{258}$$

$$= \boldsymbol{\iota}_1 \cdot \left(\boldsymbol{r} + \frac{\boldsymbol{\iota}_3 \times \boldsymbol{\delta}}{|\boldsymbol{\beta}|} \right) , \tag{259}$$

or, using (256),

$$\mathfrak{B} = \boldsymbol{\iota}_1 \cdot \left(\boldsymbol{r} + \frac{\boldsymbol{\beta} \times \boldsymbol{\delta}}{|\boldsymbol{\beta}|^2} \right) . \tag{260}$$

Finally,

$$\Sigma = \frac{|\boldsymbol{\beta}|^2}{\boldsymbol{\beta} \cdot \boldsymbol{\delta}} \boldsymbol{\iota}_3 \cdot \boldsymbol{r}$$

$$= \frac{|\boldsymbol{\beta}|^2}{\boldsymbol{\beta} \cdot \boldsymbol{\delta}} \boldsymbol{\iota}_3 \cdot \left(\boldsymbol{r} + \frac{\boldsymbol{\beta} \times \boldsymbol{\delta}}{|\boldsymbol{\beta}|^2} \right) . \tag{261}$$

It is now clear that if we choose a new system of Cartesian co-ordinates (X, Y, Z) with the origin at $(\boldsymbol{\delta} \times \boldsymbol{\beta})/|\boldsymbol{\beta}|^2$ and with the co-ordinate-axes parallel to the unit vectors $\boldsymbol{\iota}_1$, $\boldsymbol{\iota}_2$, and $\boldsymbol{\iota}_3$, respectively

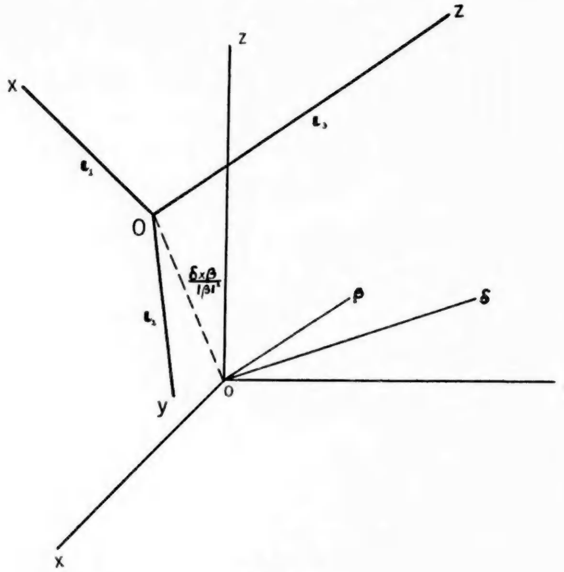

FIG. 1

(see Fig. 1), then it follows from equations (257), (260), and (261) that

$$\mathfrak{A} = Y\,; \qquad \mathfrak{B} = X\,; \qquad \Sigma = \frac{|\boldsymbol{\beta}|^2}{\boldsymbol{\beta} \cdot \boldsymbol{\delta}} Z\,. \tag{262}$$

Further, according to (248) and (262),

$$\theta = \tan^{-1} \frac{Y}{X}\,. \tag{263}$$

In this new system of co-ordinates the two integrals (252) and (253) take the forms

$$X^2 + Y^2 = \text{constant} = I_1\,, \tag{264}$$

$$Z + \frac{\boldsymbol{\beta} \cdot \boldsymbol{\delta}}{|\boldsymbol{\beta}|^2} \tan^{-1} \frac{Y}{X} = \text{constant} = I_2\,. \tag{265}$$

In the forms (264) and (265) the two integrals admit of immediate geometrical interpretations. The first integral, I_1, represents concen-

tric right-circular cylinders about the Z-axis. On the other hand, the second integral, I_2, represents right-circular helicoids, again about the Z-axis. The curve along which a right cylinder of radius $\bar{\omega}$ intersects any one of the helicoids (265) is a right-circular helix the equation of which is given parametrically by

$$X = \bar{\omega} \cos \theta \; ; \qquad Y = \bar{\omega} \sin \theta \tag{266}$$

and

$$Z = -\frac{\boldsymbol{\beta} \cdot \boldsymbol{\delta}}{|\boldsymbol{\beta}|^2} \theta + \text{constant} . \tag{267}$$

Hence, the angle of the helix, α, is given by

$$\alpha = -\tan^{-1} \frac{\boldsymbol{\beta} \cdot \boldsymbol{\delta}}{\bar{\omega} |\boldsymbol{\beta}|^2} . \tag{268}$$

Finally, according to (207),

$$\mathfrak{B}(X, Y, Z) \equiv \mathfrak{B}\left(X^2 + Y^2, Z + \frac{\boldsymbol{\beta} \cdot \boldsymbol{\delta}}{|\boldsymbol{\beta}|^2} \tan^{-1} \frac{Y}{X}\right) . \tag{269}$$

We have thus proved:

For stellar systems with differential motions the potential $\mathfrak{B}$ must necessarily be characterized by helical symmetry. The case of axial symmetry is included as a special case, namely, when $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ are orthogonal.

As we have just noted, stellar systems with axial symmetry arise when $\boldsymbol{\beta} \cdot \boldsymbol{\delta}$ is zero, i.e., when $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ are at right angles to each other.

From (269) it follows that a special case arises when $|\boldsymbol{\beta}| = 0$. This can, however, happen only when $\beta_1 = \beta_2 = \beta_3 = 0$. In this case the linear differential equation for $\mathfrak{B}$ reduces to

$$\boldsymbol{\delta} \cdot \operatorname{grad} \mathfrak{B} = 0 . \tag{270}$$

The Lagrangian subsidiary equations are

$$\frac{dx}{\delta_1} = \frac{dy}{\delta_2} = \frac{dz}{\delta_3} . \tag{271}$$

It now readily follows that

$$\mathfrak{B}(x, y, z) = \mathfrak{B}(\delta_3 x - \delta_1 z; \delta_3 y - \delta_2 z) . \tag{272}$$

Equation (272) implies that $\mathfrak{B}$ is constant along lines parallel to $\boldsymbol{\delta}$. If we choose a new system of co-ordinates (X, Y, Z) such that the Z-axis is parallel to $\boldsymbol{\delta}$, then $\partial\mathfrak{B}/\partial Z = 0$. By a further application of equations (III_4) and (IV_4) it can be shown that the system must be characterized by an axial symmetry as well. The proof of this is straightforward, and we shall omit it.

We have finally the singular case when both $|\boldsymbol{\beta}|$ and $|\boldsymbol{\delta}|$ are zero. Then $|\boldsymbol{\Delta}| = 0$, and there is no restriction on the form of $\mathfrak{B}$ arising from equation (III). Since

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix}, \tag{273}$$

it follows that in this singular case, either

$$U_0 = V_0 = W_0 = 0 \tag{274}$$

or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 . \tag{275}$$

The vanishing of the determinant in (275) is equivalent to the condition

$$abc - 2fgh - af^2 - bg^2 - ch^2 = 0 . \tag{276}$$

If we substitute for a, b, c, f, g, and h according to equations (191)–(193) and (183)–(185), we shall obtain certain restrictions on the constants occurring in our general expressions for the coefficients of the velocity ellipsoid. The case which thus arises will be similar to that we have considered in § 11 in connection with the two-dimensional problem.

We have now solved the equations for the coefficients of the velocity ellipsoid, the equations for the motions of the local centroids, and the partial differential equation for $\mathfrak{B}$. It remains to discuss the compatibility conditions (IV). But these conditions are more conveniently discussed in a suitably chosen system of co-ordinates, and we shall therefore postpone this discussion until

after we have considered stellar systems with spherical, spheroidal, and axial symmetries.

V. SPHERICAL POLAR CO-ORDINATES; SYSTEMS WITH SPHERICAL SYMMETRY

16. *The fundamental equations in spherical polar co-ordinates.*—If we choose a system of spherical polar co-ordinates (r, θ, ϕ) in defining the fundamental frame of reference, we can let

$$\lambda = r\,; \qquad \mu = \theta\,; \qquad \nu = \phi\,. \tag{277}$$

For this choice of co-ordinates

$$P = 1\,; \qquad Q = r\,; \qquad R = r\sin\theta\,. \tag{278}$$

The fundamental equations (I), (II), (III), and (IV) now take the forms

$$\left.\begin{aligned}
&\frac{\partial a}{\partial r} = 0\,, && \text{(i)}\\
&\frac{\partial b}{\partial \theta} + 2h = 0\,, && \text{(ii)}\\
&\frac{\partial c}{\partial \phi} + 2f\cos\theta + 2g\sin\theta = 0\,, && \text{(iii)}\\
&2r\frac{\partial h}{\partial r} + \frac{\partial a}{\partial \theta} - 2h = 0\,, && \text{(iv)}\\
&2r\sin\theta\frac{\partial g}{\partial r} + \frac{\partial a}{\partial \phi} - 2g\sin\theta = 0\,, && \text{(v)}\\
&2\frac{\partial h}{\partial \theta} + r\frac{\partial b}{\partial r} - 2(b-a) = 0\,, && \text{(vi)}\\
&2\sin\theta\frac{\partial f}{\partial \theta} + \frac{\partial b}{\partial \phi} + 2g\sin\theta - 2f\cos\theta = 0\,, && \text{(vii)}\\
&2\frac{\partial g}{\partial \phi} + r\sin\theta\frac{\partial c}{\partial r} - 2(c-a)\sin\theta + 2h\cos\theta = 0\,, && \text{(viii)}\\
&2\frac{\partial f}{\partial \phi} + \sin\theta\frac{\partial c}{\partial \theta} - 2(c-b)\cos\theta + 2h\sin\theta = 0\,, && \text{(ix)}\\
&2f\sin\theta + g\cos\theta - \left(r\sin\theta\frac{\partial f}{\partial r} + \sin\theta\frac{\partial g}{\partial \theta} + \frac{\partial h}{\partial \phi}\right) = 0\,, && \text{(x)}
\end{aligned}\right\} \quad (\mathrm{I}_5)$$

$$\left.\begin{aligned}
&\frac{\partial \Delta_1}{\partial r} = 0\,, && \text{(i)}\\
&\frac{\partial \Delta_2}{\partial \theta} + \Delta_1 = 0\,, && \text{(ii)}\\
&\frac{\partial \Delta_3}{\partial \phi} + \Delta_1 \sin\theta + \Delta_2 \cos\theta = 0\,, && \text{(iii)}\\
&r\frac{\partial \Delta_2}{\partial r} + \frac{\partial \Delta_1}{\partial \theta} - \Delta_2 = 0\,, && \text{(iv)}\\
&\sin\theta\,\frac{\partial \Delta_3}{\partial \theta} + \frac{\partial \Delta_2}{\partial \phi} - \Delta_3 \cos\theta = 0\,, && \text{(v)}\\
&\frac{\partial \Delta_1}{\partial \phi} + r\sin\theta\,\frac{\partial \Delta_3}{\partial r} - \Delta_3 \sin\theta = 0\,, && \text{(vi)}
\end{aligned}\right\} \qquad (\mathrm{II}_5)$$

$$\Delta_1 r \sin\theta\,\frac{\partial \mathfrak{V}}{\partial r} + \Delta_2 \sin\theta\,\frac{\partial \mathfrak{V}}{\partial \theta} + \Delta_3\,\frac{\partial \mathfrak{V}}{\partial \phi} = 0\,, \qquad (\mathrm{III}_5)$$

and

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}\begin{pmatrix} \dfrac{\partial \mathfrak{V}}{\partial r} \\[2mm] \dfrac{1}{r}\,\dfrac{\partial \mathfrak{V}}{\partial \theta} \\[2mm] \dfrac{1}{r\sin\theta}\,\dfrac{\partial \mathfrak{V}}{\partial \phi} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \dfrac{\partial \chi}{\partial r} \\[2mm] \dfrac{1}{r}\,\dfrac{\partial \chi}{\partial \theta} \\[2mm] \dfrac{1}{r\sin\theta}\,\dfrac{\partial \chi}{\partial \phi} \end{pmatrix}. \qquad (\mathrm{IV}_5)$$

17. *The solution for the coefficients of the velocity ellipsoid.*—We shall first consider the system of equations (I_5).

From (i) it is clear that a is independent of r. Differentiating equations (iv) and (v) partially with respect to r, we find

$$\frac{\partial^2 h}{\partial r^2} = 0\,; \qquad \frac{\partial^2 g}{\partial r^2} = 0\,. \qquad (279)$$

We can therefore write

$$h = h_1(\theta, \phi) r + h_2(\theta, \phi)\,, \qquad (280)$$

$$g = g_1(\theta, \phi) r + g_2(\theta, \phi)\,. \qquad (281)$$

Substituting (280) and (281) in (iv) and (v), we obtain

$$\frac{\partial a}{\partial \theta} = 2h_2(\theta, \phi) , \tag{282}$$

$$\frac{\partial a}{\partial \phi} = 2g_2(\theta, \phi) \sin \theta . \tag{283}$$

Equations (282) and (283) give rise to the integrability condition

$$\frac{\partial h_2}{\partial \phi} = \frac{\partial}{\partial \theta} (g_2 \sin \theta) . \tag{284}$$

Differentiating equations (viii) and (x) twice partially with respect to r and using equations (i) and (279), we obtain

$$\frac{\partial^3 c}{\partial r^3} = 0 ; \qquad \frac{\partial^3 f}{\partial r^3} = 0 . \tag{285}$$

We can therefore write

$$f = f_1(\theta, \phi)r^2 + f_2(\theta, \phi)r + f_3(\theta, \phi) , \tag{286}$$

$$c \sin \theta = c_1(\theta, \phi)r^2 + c_2(\theta, \phi) + c_3(\theta, \phi) . \qquad (287)[19]$$

Now, partially differentiating (vi) with respect to θ and using (280), we obtain

$$r\left(\frac{\partial^2 h_1}{\partial \theta^2} + h_1\right) + \frac{\partial^2 h_2}{\partial \theta^2} + 4h_2 = 0 . \tag{288}$$

Since h_1 and h_2 are functions of θ and ϕ only, we should have

$$\frac{\partial^2 h_1}{\partial \theta^2} + h_1 = 0 ; \qquad \frac{\partial^2 h_2}{\partial \theta^2} + 4h_2 = 0 , \tag{289}$$

or

$$h_1 = h_3(\phi) \cos \theta + h_4(\phi) \sin \theta , \tag{290}$$

$$h_2 = h_5(\phi) \cos 2\theta + h_6(\phi) \sin 2\theta . \tag{291}$$

[19] We have written "$c \sin \theta$" on the left-hand side of (287) for reasons of convenience (cf. Eq. [319]).

Hence,

$$h = (h_3 \cos \theta + h_4 \sin \theta)r + h_5 \cos 2\theta + h_6 \sin 2\theta \,. \tag{292}$$

From (282) and (291) we obtain

$$a = h_5 \sin 2\theta - h_6 \cos 2\theta + a_1(\phi) \,. \tag{293}$$

Again substituting (292) in equation (ii) and integrating, we have

$$\left.\begin{aligned} b = 2(-h_3 \sin \theta + h_4 \cos \theta)r + (-h_5 \sin 2\theta + h_6 \cos 2\theta) \\ + b_1(r, \phi) \,. \end{aligned}\right\} \tag{294}$$

Now substitute (292), (293), and (294) in equation (vi). We are left with

$$r \frac{\partial b_1}{\partial r} - 2b_1 + 2a_1(\phi) = 0 \,, \tag{295}$$

or, integrating,

$$b_1(r, \phi) = b_2(\phi)r^2 + a_1 \,. \tag{296}$$

Hence,

$$\left.\begin{aligned} b = 2(-h_3 \sin \theta + h_4 \cos \theta)r + (-h_5 \sin 2\theta + h_6 \cos 2\theta) \\ + b_2 r^2 + a_1 \,. \end{aligned}\right\} \tag{297}$$

It may be recalled that h_3, h_4, h_5, h_6, a_1, and b_2 are all functions of ϕ only.

Substituting for g, f, and b according to equations (281), (286), and (297) in equation (vii) and differentiating once with respect to θ, we obtain

$$\left.\begin{aligned} r^2\left(\frac{\partial^2 f_1}{\partial \theta^2} + f_1\right) \sin \theta + r \left(\sin \theta \frac{\partial^2 f_2}{\partial \theta^2} + f_2 \sin \theta \right. \\ \left. - \frac{\partial h_1}{\partial \phi} + \frac{\partial}{\partial \theta} [g_1 \sin \theta] \right) + \left(\frac{\partial^2 f_3}{\partial \theta^2} + f_3\right) \sin \theta = 0 \,. \end{aligned}\right\} \tag{298}$$

Hence,

$$\frac{\partial^2 f_1}{\partial \theta^2} + f_1 = 0 \,; \qquad \frac{\partial^2 f_3}{\partial \theta^2} + f_3 = 0 \tag{299}$$

and

$$\sin\theta\left(\frac{\partial^2 f_2}{\partial\theta^2}+f_2\right)=\frac{\partial h_1}{\partial\phi}-\frac{\partial}{\partial\theta}(g_1\sin\theta)\,. \tag{300}$$

Hence, according to (299),

$$f_1=f_4(\phi)\cos\theta+f_5(\phi)\sin\theta\,, \tag{301}$$

$$f_3=f_6(\phi)\cos\theta+f_7(\phi)\sin\theta\,. \tag{302}$$

Substituting for h, g, and f according to equations (280), (281), and (286) in equation (x), we obtain

$$\left.\begin{aligned}&2(f_2r+f_3)\sin\theta+(g_1r+g_2)\cos\theta\\&\qquad=rf_2\sin\theta+\sin\theta\frac{\partial}{\partial\theta}(g_1r+g_2)+\frac{\partial}{\partial\phi}(h_1r+h_2)\,.\end{aligned}\right\} \tag{303}$$

Equating the coefficients of r in (303), we obtain

$$f_2\sin\theta=-\,g_1\cos\theta+\sin\theta\frac{\partial g_1}{\partial\theta}+\frac{\partial h_1}{\partial\phi}\,. \tag{304}$$

But, according to (300),

$$f_2\sin\theta=-\sin\theta\frac{\partial^2 f_2}{\partial\theta^2}+\frac{\partial h_1}{\partial\phi}-g_1\cos\theta-\sin\theta\frac{\partial g_1}{\partial\theta}\,. \tag{305}$$

Combining (304) and (305), we find that

$$2\frac{\partial g_1}{\partial\theta}+\frac{\partial^2 f_2}{\partial\theta^2}=0\,. \tag{306}$$

Hence,

$$g_1=\gamma_1(\phi)-\frac{1}{2}\frac{\partial f_2}{\partial\theta}\,. \tag{307}$$

Equating the terms which do not involve r in (303) and using (284), we find

$$\frac{\partial g_2}{\partial\theta}=f_3=f_6\cos\theta+f_7\sin\theta\,. \tag{308}$$

Hence,

$$g_2 = f_6 \sin\theta - f_7 \cos\theta + \gamma_2(\phi)\,. \tag{309}$$

We now have, according to (283),

$$\frac{\partial a}{\partial \phi} = 2(f_6 \sin\theta - f_7\cos\theta + \gamma_2)\sin\theta\,. \tag{310}$$

But a is given by (293). We find after some reductions that

$$\left.\begin{aligned} &f_6(1-\cos 2\theta) - f_7 \sin 2\theta + 2\gamma_2 \sin\theta \\ &\qquad = -\frac{dh_6}{d\phi}\cos 2\theta + \frac{dh_5}{d\phi}\sin 2\theta + \frac{da_1}{d\phi}\,. \end{aligned}\right\} \tag{311}$$

Hence,

$$\gamma_2 = 0\,; \qquad \frac{dh_5}{d\phi} = -f_7\,; \qquad \frac{dh_6}{d\phi} = f_6\,; \qquad \frac{da_1}{d\phi} = f_6\,. \tag{312}$$

From the last two relations of (312) we infer that

$$h_6 = a_1 + \text{constant}\,. \tag{313}$$

From equation (vii) we now obtain

$$\left.\begin{aligned} 2\sin\theta&\left[(-f_4\sin\theta + f_5\cos\theta)r^2 + \frac{\partial f_2}{\partial\theta}r + (-f_6\sin\theta + f_7\cos\theta)\right] \\ &+ \left[2\left(-\frac{dh_3}{d\phi}\sin\theta + \frac{dh_4}{d\phi}\cos\theta\right)r + (f_7\sin 2\theta + f_6\cos 2\theta) + \frac{db_2}{d\phi}r^2 + f_6\right] \\ &+ 2\sin\theta\left[\left(\gamma_1 - \frac{1}{2}\frac{\partial f_2}{\partial\theta}\right)r + f_6\sin\theta - f_7\cos\theta\right] \\ &- 2\cos\theta[(f_4\cos\theta + f_5\sin\theta)r^2 + f_2 r + (f_6\cos\theta + f_7\sin\theta)] = 0\,. \end{aligned}\right\} \tag{314}$$

The terms in r^2 give

$$\frac{db_2}{d\phi} = 2f_4\,. \tag{315}$$

The terms which do not involve r are seen to cancel out. Finally, collecting together the terms in r, we have

$$\frac{\partial f_2}{\partial\theta} - 2f_2\cot\theta = -2\gamma_1 + 2\frac{dh_3}{d\phi} - 2\frac{dh_4}{d\phi}\cot\theta\,, \tag{316}$$

which is a linear equation in θ for f_2. On solving, we find that we can express the solution in the form

$$f_2 = \tfrac{1}{2}f_8(\phi)(1 - \cos 2\theta) + \gamma_1 \sin 2\theta - \frac{dh_3}{d\phi}\sin 2\theta + \frac{dh_4}{d\phi}\,. \tag{317}$$

From (307) it now follows that

$$g_1 = -\tfrac{1}{2}f_8(\phi)\sin 2\theta + \gamma_1(1 - \cos 2\theta) + \frac{dh_3}{d\phi}\cos 2\theta\,. \tag{318}$$

Substituting for c according to (287) in equation (viii), we obtain

$$\left.\begin{aligned} 2\left(\frac{\partial g_1}{\partial\phi}r + \frac{\partial g_2}{\partial\phi}\right) + rc_2(\theta,\phi) - 2[c_2(\theta,\phi)r + c_3(\theta,\phi)]& \\ + 2a\sin\theta + 2(h_1 r + h_2)\cos\theta = 0\,.& \end{aligned}\right\} \tag{319}$$

Collecting the terms in r, we obtain

$$c_2 = 2\frac{\partial g_1}{\partial\phi} + 2h_1\cos\theta\,. \tag{320}$$

Similarly the terms not involving r give

$$c_3 = \frac{\partial g_2}{\partial\phi} + a\sin\theta + h_2\cos\theta\,. \tag{321}$$

From equation (ix) we now derive

$$\left.\begin{aligned} &2\sin\theta\Big[r^2\Big\{\frac{df_4}{d\phi}\cos\theta + \frac{df_5}{d\phi}\sin\theta\Big\} + r\Big\{\tfrac{1}{2}(1-\cos 2\theta)\frac{df_8}{d\phi} + \sin 2\theta\frac{d\gamma_1}{d\phi} \\ &\quad - \sin 2\theta\frac{d^2h_3}{d\phi^2} + \frac{d^2h_4}{d\phi^2}\Big\} + \Big\{\frac{df_6}{d\phi}\cos\theta + \frac{df_7}{d\phi}\sin\theta\Big\}\Big] \\ &\quad + \sin\theta\Big[r^2\frac{\partial c_1}{\partial\theta} + r\Big\{2\frac{\partial^2 g_1}{\partial\theta\partial\phi} - 2h_1\sin\theta + 2\cos\theta\frac{\partial h_1}{\partial\theta}\Big\} \\ &\quad + \Big\{\frac{\partial^2 g_2}{\partial\theta\partial\phi} + a\cos\theta + \sin\theta\frac{\partial a}{\partial\theta} + \cos\theta\frac{\partial h_2}{\partial\theta} - h_2\sin\theta\Big\}\Big] \\ &\quad - 3\cos\theta\Big[r^2c_1 + r\Big\{2\frac{\partial g_1}{\partial\phi} + 2h_1\cos\theta\Big\} + \Big\{\frac{\partial g_2}{\partial\phi} + a\sin\theta + h_2\cos\theta\Big\}\Big] \\ &\quad + 2\cos\theta\sin\theta[r^2b_2 + 2r\{-h_3\sin\theta + h_4\cos\theta\} \\ &\quad + \{-h_5\sin 2\theta + h_6\cos 2\theta + a_1\}] + 2(h_1r + h_2)\sin^2\theta = 0\,. \end{aligned}\right\} \tag{322}$$

Collecting the terms in r^2 in (322), we find that

$$\frac{\partial c_1}{\partial\theta} - 3c_1 \cot\theta = -2b_2 \cos\theta - 2\left(\frac{df_4}{d\phi}\cos\theta + \frac{df_5}{d\phi}\sin\theta\right), \tag{323}$$

which is a linear equation in θ for c. On solving (323), we find that we can express the solution in the form

$$c_1 = c_4(\phi)\sin^3\theta + b_2\sin\theta + \sin\theta\,\frac{df_4}{d\phi} + \sin\theta\sin 2\theta\,\frac{df_5}{d\phi}. \tag{324}$$

Equation (iii) now becomes

$$\left.\begin{aligned}
&r^2\left(\frac{dc_4}{d\phi}\sin^3\theta + 2f_4\sin\theta + \frac{d^2f_4}{d\phi^2}\sin\theta + \sin 2\theta\sin\theta\,\frac{d^2f_5}{d\phi^2}\right)\\
&+ r\left(2\,\frac{\partial^2 g_1}{\partial\phi^2} + 2\cos\theta\,\frac{\partial h_1}{\partial\phi}\right) + \left(\frac{\partial^2 g_2}{\partial\phi^2} + 2g_2\sin^2\theta + \frac{\partial h_2}{\partial\phi}\cos\theta\right)\\
&+ 2\sin\theta\cos\theta\Big(r^2\{f_4\cos\theta + f_5\sin\theta\} + r\Big\{\tfrac{1}{2}(1-\cos 2\theta)f_8\\
&+ \gamma_1\sin 2\theta - \frac{dh_3}{d\phi}\sin 2\theta + \frac{dh_4}{d\phi}\Big\} + \{f_6\cos\theta + f_7\sin\theta\}\Big)\\
&+ 2\sin^2\theta\Big(r\Big\{-\tfrac{1}{2}f_8\sin 2\theta + \gamma_1(1-\cos 2\theta) + \cos 2\theta\,\frac{dh_3}{d\phi}\Big\}\\
&+ f_6\sin\theta - f_7\cos\theta\Big) = 0\,.
\end{aligned}\right\} \tag{325}$$

Collecting together the terms in r^2 in (325), we find, after some rearranging of the terms, that

$$\left.\begin{aligned}
&\left(\frac{d^2f_5}{d\phi^2} + f_5\right)\sin 2\theta + \left(-\frac{1}{2}\frac{dc_4}{d\phi} + f_4\right)\cos 2\theta + \frac{1}{2}\frac{dc_4}{d\phi} + \frac{d^2f_4}{d\phi^2}\\
&\qquad + 3f_4 = 0\,.
\end{aligned}\right\} \tag{326}$$

We should therefore have

$$\frac{d^2f_5}{d\phi^2} + f_5 = 0\,, \tag{327}$$

$$-\frac{1}{2}\frac{dc_4}{d\phi} + f_4 = 0\,, \tag{328}$$

$$\frac{1}{2}\frac{dc_4}{d\phi} + 3f_4 + \frac{d^2f_4}{d\phi^2} = 0\,. \tag{329}$$

Equations (328) and (329) combine to give

$$\frac{d^2 f_4}{d\phi^2} + 4f_4 = 0 . \tag{330}$$

Hence,

$$f_4 = \kappa_1 \cos 2\phi + \kappa_2 \sin 2\phi , \tag{331}$$

$$f_5 = \kappa_3 \cos \phi + \kappa_4 \sin \phi , \tag{332}$$

$$c_4 = \kappa_1 \sin 2\phi - \kappa_2 \cos 2\phi + \kappa_5 , \tag{333}$$

$$b_2 = \kappa_1 \sin 2\phi - \kappa_2 \cos 2\phi + \kappa_6 , \tag{334}$$

where $\kappa_1, \ldots , \kappa_6$ are arbitrary constants. The terms in (325) which do not involve r give

$$\left(\frac{d^2 f_6}{d\phi^2} + 4f_6\right) \sin \theta - \left(\frac{d^2 f_7}{d\phi^2} + f_7\right) \cos \theta = 0 . \tag{335}$$

Hence,

$$\frac{d^2 f_6}{d\phi^2} + 4f_6 = 0; \qquad \frac{d^2 f_7}{d\phi^2} + f_7 = 0 , \tag{336}$$

or, integrating,

$$f_6 = \kappa_7 \cos 2\phi + \kappa_8 \sin 2\phi , \tag{337}$$

$$f_7 = \kappa_9 \cos \phi + \kappa_{10} \sin \phi , \tag{338}$$

where $\kappa_7, \ldots , \kappa_{10}$ are again arbitrary constants. From the equations (312) we now derive

$$h_5 = -\kappa_9 \sin \phi + \kappa_{10} \cos \phi + \kappa^* , \tag{339}$$

$$h_6 = \tfrac{1}{2}(\kappa_7 \sin 2\phi - \kappa_8 \cos 2\phi) + \kappa_{11} , \tag{340}$$

$$a_1 = \tfrac{1}{2}(\kappa_7 \sin 2\phi - \kappa_8 \cos 2\phi) + \kappa_{12} , \tag{341}$$

where κ^*, κ_{11}, and κ_{12} are further constants of integration, of which κ^* is seen to be zero. For collecting the terms which do not involve r in equation (322), we find that

$$\frac{df_7}{d\phi} = h_5 , \tag{342}$$

which, according to (338) and (339), implies that $\kappa^* = 0$. The terms in r in equation (322) are found to give

$$\left(2\frac{df_8}{d\phi} + \frac{d^2h_4}{d\phi^2}\right)\sin\theta - 3\left(\frac{d^2h_3}{d\phi^2} + h_3\right)\cos\theta = 0\,. \tag{343}$$

Hence,

$$h_3 = \kappa_{13}\cos\phi + \kappa_{14}\sin\phi \tag{344}$$

and

$$\frac{dh_4}{d\phi} = \kappa_{15} - 2f_8\,. \tag{345}$$

Finally the terms in r in equation (325) reduce to

$$\left(2\frac{dh_4}{d\phi} - \frac{d^2f_8}{d\phi^2}\right)\cos\theta + 2\left(\frac{d^2\gamma_1}{d\phi^2} + \gamma_1\right)\sin\theta = 0\,. \tag{346}$$

Hence,

$$\frac{d^2\gamma_1}{d\phi^2} + \gamma_1 = 0 \tag{347}$$

and

$$\frac{d^2f_8}{d\phi^2} = 2\frac{dh_4}{d\phi}\,. \tag{348}$$

According to equations (345) and (348),

$$\frac{d^2f_8}{d\phi^2} + 4f_8 = 2\kappa_{15}\,. \tag{349}$$

Hence,

$$\gamma_1 = \kappa_{16}\cos\phi + \kappa_{17}\sin\phi\,, \tag{350}$$

$$f_8 = \kappa_{18}\cos 2\phi + \kappa_{19}\sin 2\phi + \tfrac{1}{2}\kappa_{15}\,, \tag{351}$$

$$h_4 = -\kappa_{18}\sin 2\phi + \kappa_{19}\cos 2\phi + \kappa_{20}\,. \tag{352}$$

We have now determined all the quantities in terms of r, θ, and ϕ.

Collecting our results, the solutions for a, b, c, f, g, and h can be written in the forms

$$a = h_5 \sin 2\theta - h_6 \cos 2\theta + a_1 , \tag{353}$$

$$\left.\begin{aligned} b = b_2 r^2 + 2(-h_3 \sin\theta + h_4 \cos\theta) r \\ + (-h_5 \sin 2\theta + h_6 \cos 2\theta) + a_1 , \end{aligned}\right\} \tag{354}$$

$$\left.\begin{aligned} c = \left(c_4 \sin^2\theta + \sin 2\theta \frac{df_5}{d\phi} + \frac{df_4}{d\phi} + b_2\right) r^2 + \frac{2}{\sin\theta}\left(\frac{\partial g_1}{\partial\phi} + h_1 \cos\theta\right) r \\ + \frac{1}{\sin\theta}\left(\frac{\partial g_2}{\partial\phi} + a \sin\theta + h_2 \cos\theta\right) , \end{aligned}\right\} \tag{355}$$

$$\left.\begin{aligned} f = (f_4 \cos\theta + f_5 \sin\theta) r^2 + \left\{ \tfrac{1}{2}(1 - \cos 2\theta) f_8 + \gamma_1 \sin 2\theta \right. \\ \left. - \frac{dh_3}{d\phi} \sin 2\theta + \frac{dh_4}{d\phi} \right\} r + f_6 \cos\theta + f_7 \sin\theta , \end{aligned}\right\} \tag{356}$$

$$\left.\begin{aligned} g = g_1 r + g_2 = \left\{ -\tfrac{1}{2} f_8 \sin 2\theta + \gamma_1 (1 - \cos 2\theta) + \cos 2\theta \frac{dh_3}{d\phi} \right\} r \\ + (f_6 \sin\theta - f_7 \cos\theta) , \end{aligned}\right\} \tag{357}$$

$$h = h_1 r + h_2 = (h_3 \cos\theta + h_4 \sin\theta) r + h_5 \cos 2\theta + h_6 \sin 2\theta , \tag{358}$$

where h_3, h_4, h_5, h_6, a_1, b_2, c_4, f_4, f_5, f_6, f_7, f_8, and γ_1 are all functions of ϕ only. They are given by

$$\left.\begin{aligned} h_3 &= \kappa_{13} \cos\phi + \kappa_{14} \sin\phi , \\ h_4 &= -\kappa_{18} \sin 2\phi + \kappa_{19} \cos 2\phi + \kappa_{20} , \\ h_5 &= -\kappa_9 \sin\phi + \kappa_{10} \cos\phi , \\ h_6 &= \tfrac{1}{2}(\kappa_7 \sin 2\phi - \kappa_8 \cos 2\phi) + \kappa_{11} , \\ a_1 &= \tfrac{1}{2}(\kappa_7 \sin 2\phi - \kappa_8 \cos 2\phi) + \kappa_{12} , \\ b_2 &= \kappa_1 \sin 2\phi - \kappa_2 \cos 2\phi + \kappa_6 , \\ c_4 &= \kappa_1 \sin 2\phi - \kappa_2 \cos 2\phi + \kappa_5 , \\ f_4 &= \kappa_1 \cos 2\phi + \kappa_2 \sin 2\phi , \\ f_5 &= \kappa_3 \cos\phi + \kappa_4 \sin\phi , \\ f_6 &= \kappa_7 \cos 2\phi + \kappa_8 \sin 2\phi , \\ f_7 &= \kappa_9 \cos\phi + \kappa_{10} \sin\phi , \\ f_8 &= \kappa_{18} \cos 2\phi + \kappa_{19} \sin 2\phi + \tfrac{1}{2}\kappa_{15} , \\ \gamma_1 &= \kappa_{16} \cos\phi + \kappa_{17} \sin\phi . \end{aligned}\right\} \tag{359}$$

We thus see that the general solution for the coefficients of the velocity ellipsoid involve twenty constants of integration. This is in agreement with what we have already found in solving the corresponding problem in Cartesian co-ordinates.

18. *The solution for the motions of the local centroids.*—We shall next consider the six equations (II_5) for the Δ's.

On differentiating (iv) partially with respect to r and using (i), we find that

$$\frac{\partial^2 \Delta_2}{\partial r^2} = 0 ; \tag{360}$$

in other words, Δ_2 is linear in r. Again, on differentiating (iv) with respect to θ, we find that

$$r \frac{\partial^2 \Delta_2}{\partial r \partial \theta} + \frac{\partial^2 \Delta_1}{\partial \theta^2} - \frac{\partial \Delta_2}{\partial \theta} = 0 . \tag{361}$$

On the other hand, from (i) and (ii) we easily find that

$$\frac{\partial^2 \Delta_2}{\partial r \partial \theta} = 0 . \tag{362}$$

Hence, according to (ii), (361), and (362), we have

$$\frac{\partial^2 \Delta_1}{\partial \theta^2} + \Delta_1 = 0 , \tag{363}$$

or

$$\Delta_1 = q_1(\phi) \cos \theta + q_2(\phi) \sin \theta , \tag{364}$$

since Δ_1 (according to [i]) is independent of r. From (ii) and (364) we now obtain

$$\Delta_2 = -q_1 \sin \theta + q_2 \cos \theta + q_3(\phi) r + q_4(\phi) , \tag{365}$$

since Δ_2 is linear in r. But, according to (iv),

$$\Delta_2 = r \frac{\partial \Delta_2}{\partial r} + \frac{\partial \Delta_1}{\partial \theta} . \tag{366}$$

from (364), (365), and (366) we find that $q_4 = 0$. Hence,

$$\Delta_2 = -q_1 \sin\theta + q_2 \cos\theta + q_3 r\,, \tag{367}$$

where it will be recalled that q_1, q_2, and q_3 are all functions of ϕ only.

Substituting in equation (iii) for Δ_1 and Δ_2 according to (364) and (367), we find that

$$\frac{\partial\Delta_3}{\partial\phi} = -(q_2 + q_3 r \cos\theta)\,. \tag{368}$$

Differentiating (vi) partially with respect to ϕ and using (364) and (368), we obtain

$$\cos\theta\,\frac{d^2 q_1}{d\phi^2} + \sin\theta\left(\frac{d^2 q_2}{d\phi^2} + q_2\right) = 0\,, \tag{369}$$

or

$$\frac{d^2 q_1}{d\phi^2} = 0\,; \qquad \frac{d^2 q_2}{d\phi^2} + q_2 = 0\,. \tag{370}$$

Hence,

$$q_1 = k^*\phi + k_1\,; \qquad q_2 = k_2 \cos\phi + k_3 \sin\phi\,, \tag{371}$$

where k^*, k_1, k_2, and k_3 are constants. Similarly, on differentiating (v) partially with respect to ϕ and using (367), (368), and (370), we find

$$\frac{d^2 q_3}{d\phi^2} + q_3 = 0\,, \tag{372}$$

or

$$q_3 = k_4 \cos\phi + k_5 \sin\phi\,. \tag{373}$$

Hence, according to equations (364), (367), (371), and (373),

$$\Delta_1 = (k^*\phi + k_1)\cos\theta + (k_2\cos\phi + k_3\sin\phi)\sin\theta\,, \tag{374}$$

$$\left.\begin{aligned}\Delta_2 = -(k^*\phi + k_1)\sin\theta + (k_2\cos\phi + k_3\sin\phi)\cos\theta \\ + r(k_4\cos\phi + k_5\sin\phi)\,.\end{aligned}\right\} \tag{375}$$

Again, from (368), (371), and (373) we now have

$$\frac{\partial \Delta_3}{\partial \phi} = -(k_2 \cos \phi + k_3 \sin \phi) - (k_4 \cos \phi + k_5 \sin \phi) r \cos \theta , \tag{376}$$

or, integrating,

$$\left.\begin{aligned} \Delta_3 = &-(k_2 \sin \phi - k_3 \cos \phi) \\ &\qquad - (k_4 \sin \phi - k_5 \cos \phi) r \cos \theta + \delta_3(r, \theta) . \end{aligned}\right\} \tag{377}$$

Substituting for Δ_1 and Δ_3 (according to [374] and [377]) in equation (vi), we find that

$$r \frac{\partial \delta_3}{\partial r} - \delta_3 = -k^* \frac{\cos \theta}{\sin \theta} . \tag{378}$$

Hence,

$$\delta_3 = r \delta_4(\theta) + k^* \frac{\cos \theta}{\sin \theta} . \tag{379}$$

Finally, on substituting for Δ_2 and Δ_3 in equation (v), we find after some reductions that

$$k^* = 0, \qquad \delta_4 = k_6 \sin \theta . \tag{380}$$

Hence, our solutions are

$$\Delta_1 = k_1 \cos \theta + (k_2 \cos \phi + k_3 \sin \phi) \sin \theta , \tag{381}$$

$$\left.\begin{aligned} \Delta_2 = &-k_1 \sin \theta + (k_2 \cos \phi + k_3 \sin \phi) \cos \theta \\ &\qquad + r(k_4 \cos \phi + k_5 \sin \phi) , \end{aligned}\right\} \tag{382}$$

$$\left.\begin{aligned} \Delta_3 = &-(k_2 \sin \phi - k_3 \cos \phi) - r(k_4 \sin \phi - k_5 \cos \phi) \cos \theta \\ &\qquad + k_6 r \sin \theta . \end{aligned}\right\} \tag{383}$$

19. *Stellar systems with spherical symmetry: the dynamics of globular clusters with differential rotations.*—We shall say that a stellar system is characterized by spherical symmetry if

$$\frac{\partial \mathfrak{V}}{\partial \theta} = \frac{\partial \mathfrak{V}}{\partial \phi} = 0 . \tag{384}$$

Equation (III_5) now reduces to

$$\Delta_1 \frac{d\mathfrak{B}}{dr} = 0 . \tag{385}$$

Hence, if $d\mathfrak{B}/dr \neq 0$ (the only nontrivial case),

$$\Delta_1 \equiv 0 . \tag{386}$$

According to (381), this implies that

$$k_1 = k_2 = k_3 = 0 . \tag{387}$$

Hence, for systems with spherical symmetry we should have

$$\Delta_1 = 0 , \tag{388}$$

$$\Delta_2 = r(k_4 \cos\phi + k_5 \sin\phi) , \tag{389}$$

$$\Delta_3 = -r(k_4 \sin\phi - k_5 \cos\phi)\cos\theta + k_6 r \sin\theta . \tag{390}$$

We consider next the equations (IV_5). For systems with spherical symmetry they reduce to

$$\left.\begin{aligned} -\frac{1}{2}\frac{\partial\chi}{\partial r} &= a\frac{d\mathfrak{B}}{dr} , \\ -\frac{1}{2}\frac{\partial\chi}{\partial\theta} &= rh\frac{d\mathfrak{B}}{dr} , \\ -\frac{1}{2}\frac{\partial\chi}{\partial\phi} &= rg\sin\theta\frac{d\mathfrak{B}}{dr} . \end{aligned}\right\} \tag{391}$$

Equations (391) lead to the following integrability conditions:

$$\frac{\partial}{\partial\theta}\left(a\frac{d\mathfrak{B}}{dr}\right) = \frac{\partial}{\partial r}\left(rh\frac{d\mathfrak{B}}{dr}\right) , \tag{392}$$

$$\frac{\partial}{\partial\phi}\left(rh\frac{d\mathfrak{B}}{dr}\right) = \frac{\partial}{\partial\theta}\left(rg\sin\theta\frac{d\mathfrak{B}}{dr}\right) , \tag{393}$$

$$\frac{\partial}{\partial r}\left(rg\sin\theta\frac{d\mathfrak{B}}{dr}\right) = \frac{\partial}{\partial\phi}\left(a\frac{d\mathfrak{B}}{dr}\right) . \tag{394}$$

These equations are easily seen to be equivalent to

$$\left(\frac{\partial a}{\partial \theta} - r\frac{\partial h}{\partial r} - h\right)\frac{d\mathfrak{B}}{dr} = rh\frac{d^2\mathfrak{B}}{dr^2}\,, \tag{395}$$

$$\left(\frac{\partial a}{\partial \phi} - g\sin\theta - r\sin\theta\frac{\partial g}{\partial r}\right)\frac{d\mathfrak{B}}{dr} = rg\sin\theta\frac{d^2\mathfrak{B}}{dr^2}\,, \tag{396}$$

$$\frac{\partial h}{\partial \phi} = \frac{\partial}{\partial \theta}(g\sin\theta)\,. \tag{397}$$

These integrability conditions will be satisfied identically, i.e., with no restriction on $\mathfrak{B}(r)$ if, and only if,

$$h = g = 0\,, \qquad \frac{\partial a}{\partial \theta} = \frac{\partial a}{\partial \phi} = 0\,. \tag{398}$$

According to equations (353), (357), (358), and (359), equations (398) imply that

$$\kappa_{7,\,8,\,9,\,10,\,11,\,13,\,14,\,15,\,16,\,17,\,18,\,19,\,20} = 0\,. \tag{399}$$

Hence, according to equations (353)–(359), we now have

$$\left.\begin{aligned}
a &= \kappa_{12}\,,\\
b &= b_2 r^2 + \kappa_{12}\,,\\
c &= \left(c_4\sin^2\theta + \sin 2\theta\frac{df_5}{d\phi} + \frac{df_4}{d\phi} + b_2\right)r^2 + \kappa_{12}\,,\\
f &= (f_4\cos\theta + f_5\sin\theta)r^2; \qquad g = h = 0\,,
\end{aligned}\right\} \tag{400}$$

where

$$\left.\begin{aligned}
b_2 &= \kappa_1\sin 2\phi - \kappa_2\cos 2\phi + \kappa_6\,,\\
c_4 &= \kappa_1\sin 2\phi - \kappa_2\cos 2\phi + \kappa_5\,,\\
f_4 &= \kappa_1\cos 2\phi + \kappa_2\sin 2\phi\,,\\
f_5 &= \kappa_3\cos\phi + \kappa_4\sin\phi\,.
\end{aligned}\right\} \tag{401}$$

The solution is thus seen to involve seven arbitrary constants. However, by a suitable choice of the direction of the x-axis we can arrange that the equations (401) take the simpler forms

$$b_2 = \kappa_1 \sin 2\phi + \kappa_6 ; \qquad c_4 = \kappa_1 \sin 2\phi + \kappa_5 ; \qquad f_4 = \kappa_1 \cos 2\phi \tag{402}$$

and

$$f_5 = \kappa_3 \cos \phi + \kappa_4 \sin \phi , \tag{403}$$

in which case the equations (400) reduce to

$$\left.\begin{aligned} a &= \kappa_{12} , \\ b &= (\kappa_1 \sin 2\phi + \kappa_6) r^2 + \kappa_{12} , \\ c &= [\kappa_5 \sin^2 \theta - \kappa_1 \sin 2\phi \cos^2 \theta \\ &\qquad - (\kappa_3 \sin \phi - \kappa_4 \cos \phi) \sin 2\theta + \kappa_6] r^2 + \kappa_{12} , \\ f &= [\kappa_1 \cos 2\phi \cos \theta + (\kappa_3 \cos \phi + \kappa_4 \sin \phi) \sin \theta] r^2 , \\ g &= h = 0 . \end{aligned}\right\} \tag{404}$$

We thus see that when the integrability conditions (395), (396), and (397) are satisfied identically, the solution for the coefficients of the velocity ellipsoid for stellar systems with spherical symmetry involves six arbitrary (nontrivial) constants.

From equations (400) or (404) it is clear that one of the principal axes of the velocity ellipsoid is in the radial direction; there is, however, a deviation of the vertex in the transverse plane.

Let us now denote the components of the motion of the local centroid along the principal directions at (r, θ, ϕ) by P_0, Θ_0, and Φ_0. By definition

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \begin{pmatrix} P_0 \\ \Theta_0 \\ \Phi_0 \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} . \tag{405}$$

For the case under consideration equation (405) reduces to (cf. Eqs. [388], [389], [390], and [400])

$$a P_0 = 0 , \tag{406}$$

$$b \Theta_0 + f \Phi_0 = \Delta_2 , \tag{407}$$

$$f \Theta_0 + c \Phi_0 = \Delta_3 . \tag{408}$$

According to (406),

$$P_0 = 0 \qquad (a \neq 0) \,. \quad (409)$$

Hence, when the integrability conditions are satisfied identically, there are no dilatational motions of the local centroids. However, according to equations (407) and (408), the motions of the local centroids cannot be described as pure "rotations" about an axis of symmetry.

We shall next examine whether there are certain special forms for $\mathfrak{B}(r)$ for which the integrability conditions (395)–(397) can be satisfied:

Using equations (280), (281), (282), (283), and (284), the integrability conditions are easily reduced to the forms

$$\frac{d^2\mathfrak{B}}{dr^2}\Big/\frac{d\mathfrak{B}}{dr} = \frac{h_2 - 2rh_1}{r(h_1 r + h_2)} \,, \qquad (410)$$

$$\frac{d^2\mathfrak{B}}{dr^2}\Big/\frac{d\mathfrak{B}}{dr} = \frac{g_2 - 2rg_1}{r(g_1 r + g_2)} \,, \qquad (411)$$

$$\frac{\partial h_1}{\partial \phi} = \frac{\partial}{\partial \theta}(g_1 \sin\theta) \,, \qquad (412)$$

where it will be recalled that h_1, h_2, g_1, and g_2 are functions of θ and ϕ.

Since $\mathfrak{B}$ is assumed to be a function of r only, it is clear that the right-hand sides of the equations (410) and (411) should be independent of θ and ϕ. This can be arranged in one of the following two ways. Either

$$h_1 = g_1 = 0 \,, \qquad h_2,\ g_2 \neq 0 \,; \qquad \frac{d^2\mathfrak{B}}{dr^2} = \frac{1}{r}\frac{d\mathfrak{B}}{dr} \qquad (\text{case } \alpha_1) \quad (413)$$

or

$$h_2 = g_2 = 0 \,, \qquad h_1,\ g_1 \neq 0 \,; \qquad \frac{d^2\mathfrak{B}}{dr^2} = -\frac{2}{r}\frac{d\mathfrak{B}}{dr} \qquad (\text{case } \alpha_2) \,. \quad (414)$$

Let us first consider case α_1. According to (413), we now have

$$\frac{d\mathfrak{B}}{dr} = \text{constant } r \,, \qquad (415)$$

a field of force which will exist inside a homogeneous spherical distribution of matter. In this case equation (412) is identically satisfied. Hence, according to equations (357) and (358), we should have

$$h_3 = h_4 = f_8 = \gamma_1 = 0\,. \tag{416}$$

From equation (359) it now follows that

$$\kappa_{13,\,14,\,15,\,16,\,17,\,18,\,19,\,20} = 0\,. \tag{417}$$

The solution for the coefficients of the velocity ellipsoid appropriate to the present case can be written down from the general solution obtained in § 17 by setting the eight constants of integration (417) equal to zero.

Let us next consider the case a_2. According to (414), we now have

$$\frac{d\mathfrak{V}}{dr} = \frac{\text{constant}}{r^2}\,, \tag{418}$$

which represents an inverse square law of force. According to equation (414) we now have $h_2 = g_2 = 0$. Substituting for h_1 and g_1 from (357) and (358) in equation (412), we obtain after some minor transformations

$$\left.\begin{aligned} -\tfrac{3}{4}f_8 \sin 3\theta + \frac{3}{2}\left(\frac{dh_3}{d\phi} - \gamma_1\right)\cos 3\theta + \frac{3}{2}\left(\gamma_1 - \frac{dh_3}{d\phi}\right)\cos\theta \\ + \left(\tfrac{1}{4}f_8 - \frac{dh_4}{d\phi}\right)\sin\theta = 0\,. \end{aligned}\right\} \tag{419}$$

We should therefore have

$$f_8 = \frac{dh_4}{d\phi} = 0\,; \qquad \gamma_1 = \frac{dh_3}{d\phi}\,. \tag{419'}$$

Combining (414) and (419′), we have

$$h_5 = h_6 = f_6 = f_7 = f_8 = \frac{dh_4}{d\phi} = 0\,; \qquad \gamma_1 = \frac{dh_3}{d\phi}\,. \tag{420}$$

From equations (357), (358), and (359) we find that (420) implies

$$\kappa_{7,\,8,\,9,\,10,\,11,\,15,\,18,\,19} = 0\,; \qquad \kappa_{16} = \kappa_{14}\,; \qquad \kappa_{17} = -\kappa_{13}\,. \tag{421}$$

Again, the solution for the coefficients of the velocity ellipsoid appropriate to the case a_2 can be written down from the general solution obtained in § 17. For the case under consideration (Eq. [418]) the solution is seen to involve ten constants of integration.

We have thus shown that for the two special cases

$$\mathfrak{V} = \text{constant } r^2 \; ; \qquad \mathfrak{V} = \frac{\text{constant}}{r} \, , \tag{422}$$

stellar systems with spherical symmetry will be characterized by deviations of the vertex with respect to all the three principal directions. Further, the motions of the local centroids will have dilatational components as well.

This completes our discussion of stellar systems with spherical symmetry.

VI. SPHEROIDAL CO-ORDINATES

20. *The fundamental equations in oblate spheroidal co-ordinates.*—A system of oblate spheroidal co-ordinates (λ, μ, ν) is given by

$$\left.\begin{aligned} x &= \cosh \lambda \cos \mu \cos \nu \, , \\ y &= \cosh \lambda \cos \mu \sin \nu \, , \\ z &= \sinh \lambda \sin \mu \, . \end{aligned}\right\} \tag{423}$$

From (423) we derive that

$$\frac{x^2 + y^2}{\cosh^2 \lambda} + \frac{z^2}{\sinh^2 \lambda} = 1 \tag{424}$$

and

$$\frac{x^2 + y^2}{\cos^2 \mu} - \frac{z^2}{\sin^2 \mu} = 1 \, . \tag{425}$$

For a given λ, equation (424) defines an oblate spheroid with major and minor axes given by $\cosh \lambda$ and $\sinh \lambda$, respectively. In the same way, for a given μ, equation (425) defines a hyperbolid of two sheets. Further, it is clear that ν has the same geometrical meaning as the "azimuthal" angle. The spheroids (424), the hyper-

bolids (425), and the planes ν=constant provide the basis for an orthogonal system of co-ordinates. The half-distance between the foci of the confocal spheroids (424) is seen to be unity. This implies, however, no loss of generality, since we can always choose our unit of length to be the distance between the two foci.

For the system of co-ordinates (423) it is found that

$$P^2 = Q^2 = \cosh^2\lambda - \cos^2\mu = \tfrac{1}{2}(\cosh 2\lambda - \cos 2\mu) \tag{426}$$

and

$$R = \cosh\lambda\cos\mu\,. \tag{427}$$

We shall consider the fundamental differential equations in the forms (I_1), (II_1), (III_1), and (IV_1) obtained in Part I. The system of partial differential equations in our present system of co-ordinates is

$$\left.\begin{aligned}
&\tfrac{1}{2}(\cosh 2\lambda - \cos 2\mu)\frac{\partial A}{\partial\lambda} + A\sinh 2\lambda + H\sin 2\mu = 0\,, &&\text{(i)}\\
&\tfrac{1}{2}(\cosh 2\lambda - \cos 2\mu)\frac{\partial B}{\partial\mu} + H\sinh 2\lambda + B\sin 2\mu = 0\,, &&\text{(ii)}\\
&\tfrac{1}{2}\cosh\lambda\cos\mu\frac{\partial C}{\partial\nu} + G\sinh\lambda\cos\mu - F\cosh\lambda\sin\mu = 0\,, &&\text{(iii)}\\
&\frac{\partial A}{\partial\mu} + 2\frac{\partial H}{\partial\lambda} + \frac{2H\sinh 2\lambda + 2B\sin 2\mu}{\cosh 2\lambda - \cos 2\mu} = 0\,, &&\text{(iv)}\\
&\frac{\cosh 2\lambda - \cos 2\mu}{2\cosh^2\lambda\cos^2\mu}\frac{\partial A}{\partial\nu} + 2\frac{\partial G}{\partial\lambda} + \frac{2G\sinh 2\lambda + 2F\sin 2\mu}{\cosh 2\lambda - \cos 2\mu} = 0\,, &&\text{(v)}\\
&\frac{\partial B}{\partial\lambda} + 2\frac{\partial H}{\partial\mu} + \frac{2A\sinh 2\lambda + 2H\sin 2\mu}{\cosh 2\lambda - \cos 2\mu} = 0\,, &&\text{(vi)}\\
&\frac{\cosh 2\lambda - \cos 2\mu}{2\cosh^2\lambda\cos^2\mu}\frac{\partial B}{\partial\nu} + 2\frac{\partial F}{\partial\mu} + \frac{2G\sinh 2\lambda + 2F\sin 2\mu}{\cosh 2\lambda - \cos 2\mu} = 0\,, &&\text{(vii)}\\
&\frac{2\cosh^2\lambda\cos^2\mu}{\cosh 2\lambda - \cos 2\mu}\frac{\partial C}{\partial\lambda} + 2\frac{\partial G}{\partial\nu} + 2\frac{\sinh\lambda}{\cosh\lambda}A - 2\frac{\sin\mu}{\cos\mu}H = 0\,, &&\text{(viii)}\\
&\frac{2\cosh^2\lambda\cos^2\mu}{\cosh 2\lambda - \cos 2\mu}\frac{\partial C}{\partial\mu} + 2\frac{\partial F}{\partial\nu} + 2\frac{\sinh\lambda}{\cosh\lambda}H - 2\frac{\sin\mu}{\cos\mu}B = 0\,, &&\text{(ix)}\\
&\frac{\partial F}{\partial\lambda} + \frac{\partial G}{\partial\mu} + \frac{\cosh 2\lambda - \cos 2\mu}{2\cosh^2\lambda\cos^2\mu}\frac{\partial H}{\partial\nu} = 0\,, &&\text{(x)}
\end{aligned}\right\}\ (I_6)$$

$$\left.\begin{aligned}
&\frac{\partial \xi}{\partial \lambda} + \frac{\xi \sinh 2\lambda + \eta \sin 2\mu}{\cosh 2\lambda - \cos 2\mu} = 0\,, && \text{(i)}\\
&\frac{\partial \eta}{\partial \mu} + \frac{\xi \sinh 2\lambda + \eta \sin 2\mu}{\cosh 2\lambda - \cos 2\mu} = 0\,, && \text{(ii)}\\
&\frac{\partial \zeta}{\partial \nu} + \frac{\sinh \lambda}{\cosh \lambda}\,\xi - \frac{\sin \mu}{\cos \mu}\,\eta = 0\,, && \text{(iii)}\\
&\frac{\partial \xi}{\partial \mu} + \frac{\partial \eta}{\partial \lambda} = 0\,, && \text{(iv)}\\
&(\cosh 2\lambda - \cos 2\mu)\,\frac{\partial \eta}{\partial \nu} + 2\cosh^2\lambda \cos^2\mu\,\frac{\partial \zeta}{\partial \mu} = 0\,, && \text{(v)}\\
&(\cosh 2\lambda - \cos 2\mu)\,\frac{\partial \xi}{\partial \nu} + 2\cosh^2\lambda \cos^2\mu\,\frac{\partial \zeta}{\partial \lambda} = 0\,, && \text{(vi)}
\end{aligned}\right\} \qquad (\mathrm{II}_6)$$

$$\xi\,\frac{\partial \mathfrak{V}}{\partial \lambda} + \eta\,\frac{\partial \mathfrak{V}}{\partial \mu} + \zeta\,\frac{\partial \mathfrak{V}}{\partial \nu} = 0\,, \qquad (\mathrm{III}_6)$$

and

$$\begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix}\begin{pmatrix} \dfrac{\partial \mathfrak{V}}{\partial \lambda} \\ \dfrac{\partial \mathfrak{V}}{\partial \mu} \\ \dfrac{\partial \mathfrak{V}}{\partial \nu} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \dfrac{1}{P^2}\dfrac{\partial \chi}{\partial \lambda} \\ \dfrac{1}{P^2}\dfrac{\partial \chi}{\partial \mu} \\ \dfrac{1}{R^2}\dfrac{\partial \chi}{\partial \nu} \end{pmatrix}. \qquad (\mathrm{IV}_6)$$

21. *The solution for the coefficients of the velocity ellipsoid.*—We shall first consider the system of equations (I_6).

Combining equations (i) and (vi) and (ii) and (iv), we obtain, respectively,

$$\frac{\partial A}{\partial \lambda} = \frac{\partial B}{\partial \lambda} + 2\,\frac{\partial H}{\partial \mu} \qquad (428)$$

and

$$\frac{\partial B}{\partial \mu} = \frac{\partial A}{\partial \mu} + 2\,\frac{\partial H}{\partial \lambda}\,. \qquad (429)$$

We can write the equations (428) and (429) alternatively in the forms

$$\left.\begin{aligned}\frac{\partial}{\partial\lambda}(A-B) &= 2\frac{\partial H}{\partial\mu}\,,\\ \frac{\partial}{\partial\mu}(A-B) &= -2\frac{\partial H}{\partial\lambda}\,.\end{aligned}\right\}\tag{430}$$

From the equations (430) we easily derive that

$$\frac{\partial^2}{\partial\lambda^2}(A-B)+\frac{\partial^2}{\partial\mu^2}(A-B)=0\,,\tag{431}$$

$$\frac{\partial^2 H}{\partial\lambda^2}+\frac{\partial^2 H}{\partial\mu^2}=0\,.\tag{432}$$

The partial differential equations (431) and (432) are of the standard forms and can be solved. We can thus write

$$H=H_1(i\lambda+\mu,\nu)+H_2(i\lambda-\mu,\nu)\,.\tag{433}$$

From the equations (430) it now follows that

$$A-B=-2iH_1(i\lambda+\mu,\nu)+2iH_2(i\lambda-\mu,\nu)\,.\tag{433'}$$

Introduce the two variables p and q, defined by

$$p=i\lambda+\mu;\qquad q=i\lambda-\mu\,,\tag{434}$$

or

$$\lambda=-\frac{i}{2}(p+q)\,;\qquad \mu=\tfrac{1}{2}(p-q)\,.\tag{435}$$

We can now write

$$H=H_1(p,\nu)+H_2(q,\nu)\,,\tag{436}$$

$$A-B=-2iH_1(p,\nu)+2iH_2(q,\nu)\,.\tag{437}$$

Now, equations (i) and (ii) can be written in the forms

$$\frac{\partial}{\partial\lambda}\{\tfrac{1}{2}(\cosh 2\lambda - \cos 2\mu)A\} = -H \sin 2\mu\,, \tag{438}$$

$$\frac{\partial}{\partial\mu}\{\tfrac{1}{2}(\cosh 2\lambda - \cos 2\mu)B\} = -H \sinh 2\lambda\,. \tag{439}$$

Since

$$\left.\begin{aligned}\tfrac{1}{2}(\cosh 2\lambda - \cos 2\mu) = -\sin(\lambda i + \mu)\sin(\lambda i - \mu)\\ = -\sin p \sin q\,,\end{aligned}\right\} \tag{440}$$

$$\sin 2\mu = \sin(p-q)\,, \tag{441}$$

and

$$\sinh 2\lambda = -i\sin 2\lambda i = -i\sin(p+q)\,, \tag{442}$$

we can re-write (438) and (439) as

$$\frac{\partial}{\partial\lambda}(A\sin p\sin q) = H\sin(p-q)\,, \tag{443}$$

$$\frac{\partial}{\partial\mu}(B\sin p\sin q) = -iH\sin(p+q)\,. \tag{444}$$

But according to (434) and (435),

$$\frac{\partial}{\partial\lambda} = i\left(\frac{\partial}{\partial p}+\frac{\partial}{\partial q}\right); \qquad \frac{\partial}{\partial\mu} = \frac{\partial}{\partial p}-\frac{\partial}{\partial q} \tag{445}$$

and

$$\frac{\partial}{\partial p} = \frac{1}{2}\left(-i\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\mu}\right); \qquad \frac{\partial}{\partial q} = -\frac{1}{2}\left(i\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\mu}\right). \tag{445'}$$

From equations (443), (444), and (445) we obtain, after some transformations,

$$\left.\begin{aligned}\frac{\partial}{\partial p}\{(A-B)\sin p\sin q\}+\frac{\partial}{\partial q}\{(A+B)\sin p\sin q\}\\ = 2iH\cos p\sin q\,,\end{aligned}\right\} \tag{446}$$

$$\left.\begin{aligned}\frac{\partial}{\partial p}\{(A+B)\sin p\sin q\}+\frac{\partial}{\partial q}\{(A-B)\sin p\sin q\}\\ = -2iH\sin p\cos q\,.\end{aligned}\right\} \tag{447}$$

Differentiate equations (446) and (447) partially with respect to p and q, respectively, and subtract. Using equations (436) and (437), and after some simplifications, we obtain

$$\left\{ \sin p \frac{\partial^2 H_1}{\partial p^2} + 3 \cos p \frac{\partial H_1}{\partial p} - 2H_1 \sin p \right\} \sin q + \left\{ \sin q \frac{\partial^2 H_2}{\partial q^2} + 3 \cos q \frac{\partial H_2}{\partial q} - 2H_2 \sin q \right\} \sin p = 0 . \tag{448}$$

Since H_1 and H_2 are independent of q and p, respectively, it is clear that we should have

$$\sin p \frac{\partial^2 H_1}{\partial p^2} + 3 \cos p \frac{\partial H_1}{\partial p} - 2H_1 \sin p = 2h_0 \sin p \tag{449}$$

and

$$\sin q \frac{\partial^2 H_2}{\partial q^2} + 3 \cos q \frac{\partial H_2}{\partial q} - 2H_2 \sin q = -2h_0 \sin q , \tag{450}$$

where h_0 is a function of ν only. Equation (449) is readily seen to be equivalent to

$$\frac{\partial^2}{\partial p^2} (H_1 \sin p) + \frac{\partial}{\partial p} (H_1 \cos p) = -2h_0 \frac{\partial}{\partial p} (\cos p) . \tag{451}$$

On integrating (451), we obtain

$$\frac{\partial}{\partial p} (H_1 \sin p) + H_1 \cos p = -2h_0 \cos p + 2h_1 , \tag{452}$$

where h_1 is a function of ν only. Equation (452) can be transformed into

$$\sin p \frac{\partial}{\partial p} (H_1 + h_0) + 2(H_1 + h_0) \cos p = 2h_1 , \tag{453}$$

which is a linear equation for $H_1 + h_0$. On solving (453), we find

$$H_1 + h_0 = \frac{2}{\sin^2 p} (h_3 - h_1 \cos p) , \tag{454}$$

where h_3 is again a function of ν only. In the same way we obtain from (450)

$$H_2 - h_0 = \frac{2}{\sin^2 q}(h_4 - h_2 \cos q) \,, \tag{455}$$

where h_4 and h_2 are functions of ν only.

From equations (446), (436), and (437) we derive

$$\frac{\partial}{\partial q}\{(A + B) \sin p \sin q\} = 2i\left(\sin p \frac{\partial H_1}{\partial p} + 2H_1 \cos p\right)\sin q \,, \tag{456}$$

or, according to (452),

$$\frac{\partial}{\partial q}\{(A + B) \sin p \sin q\} = 4i(h_1 - h_0 \cos p) \sin q \,. \tag{457}$$

Similarly, from (447) we obtain

$$\frac{\partial}{\partial p}\{(A + B) \sin p \sin q\} = -4i(h_2 + h_0 \cos q) \sin p \,. \tag{458}$$

Equations (457) and (458) satisfy the necessary integrability condition, i.e.,

$$\sin q \frac{\partial}{\partial p}(h_1 - h_0 \cos p) = -\sin p \frac{\partial}{\partial q}(h_2 + h_0 \cos q) \,. \tag{459}$$

Hence, from (457) and (458) we obtain

$$(A + B) \sin p \sin q = 4i(h_0 \cos p \cos q - h_1 \cos q + h_2 \cos p + h_5) \,, \tag{460}$$

where h_5 is a function of ν only. Equations (436), (437), (454), (455), and (460) suffice to determine the dependence of A, B, and H on λ and μ.

Combining equations (v) and (vii), we obtain

$$\frac{\cosh 2\lambda - \cos 2\mu}{2 \cosh^2 \lambda \cos^2 \mu} \frac{\partial}{\partial \nu}(A - B) = 2\frac{\partial F}{\partial \mu} - 2\frac{\partial G}{\partial \lambda} \,, \tag{461}$$

or, according to (437),

$$\frac{\cosh 2\lambda - \cos 2\mu}{2\cosh^2\lambda\cos^2\mu}\frac{\partial}{\partial\nu}(H_1 - H_2) = i\frac{\partial F}{\partial\mu} - i\frac{\partial G}{\partial\lambda}. \tag{462}$$

From equation (x), on the other hand, we have

$$\frac{\cosh 2\lambda - \cos 2\mu}{2\cosh^2\lambda\cos^2\mu}\frac{\partial}{\partial\nu}(H_1 + H_2) = -\frac{\partial F}{\partial\lambda} - \frac{\partial G}{\partial\mu}. \tag{463}$$

From (462) and (463) we readily obtain

$$\frac{\cosh 2\lambda - \cos 2\mu}{\cosh^2\lambda\cos^2\mu}\frac{\partial H_1}{\partial\nu} = i\left(\frac{\partial F}{\partial\mu} + i\frac{\partial F}{\partial\lambda}\right) - \left(i\frac{\partial G}{\partial\lambda} + \frac{\partial G}{\partial\mu}\right) \tag{464}$$

and

$$\frac{\cosh 2\lambda - \cos 2\mu}{\cosh^2\lambda\cos^2\mu}\frac{\partial H_2}{\partial\nu} = -i\left(\frac{\partial F}{\partial\mu} - i\frac{\partial F}{\partial\lambda}\right) + \left(i\frac{\partial G}{\partial\lambda} - \frac{\partial G}{\partial\mu}\right), \tag{465}$$

or, using (445′),

$$-\frac{\cosh 2\lambda - \cos 2\mu}{2\cosh^2\lambda\cos^2\mu}\frac{\partial H_1}{\partial\nu} = \frac{\partial}{\partial q}(iF - G), \tag{466}$$

$$-\frac{\cosh 2\lambda - \cos 2\mu}{2\cosh^2\lambda\cos^2\mu}\frac{\partial H_2}{\partial\nu} = \frac{\partial}{\partial p}(iF + G). \tag{467}$$

It is readily shown that

$$\frac{\cosh 2\lambda - \cos 2\mu}{2\cosh^2\lambda\cos^2\mu} = -4\frac{\sin p\sin q}{(\cos p + \cos q)^2}. \tag{468}$$

We can now re-write the equations (466) and (467) in the forms

$$\frac{\partial\phi}{\partial q} = \frac{4\sin p\sin q}{(\cos p + \cos q)^2}\frac{\partial H_1}{\partial\nu}, \tag{469}$$

$$\frac{\partial\psi}{\partial p} = \frac{4\sin p\sin q}{(\cos p + \cos q)^2}\frac{\partial H_2}{\partial\nu}, \tag{470}$$

where we have used ϕ and ψ to denote

$$\phi = iF - G \,; \qquad \psi = iF + G \,. \tag{471}$$

Remembering that H_1 and H_2 are independent of q and p, respectively, we obtain, on integrating the equations (469) and (470),

$$\phi = \frac{4 \sin p}{\cos p + \cos q} \frac{\partial H_1}{\partial \nu} + \phi_1(p, \nu) \,, \tag{472}$$

$$\psi = \frac{4 \sin q}{\cos p + \cos q} \frac{\partial H_2}{\partial \nu} + \psi_1(q, \nu) \,, \tag{473}$$

where ϕ_1 and ψ_1 are independent of q and p, respectively.

Adding equations (v) and (vii), we obtain, after several transformations,

$$\left.\begin{aligned} &\frac{1}{\sin q} \frac{\partial}{\partial q} (\psi \sin q) - \frac{1}{\sin p} \frac{\partial}{\partial p} (\phi \sin p) \\ &\qquad = -\frac{2i}{(\cos p + \cos q)^2} \frac{\partial}{\partial \nu} \{(A + B) \sin p \sin q\} \,. \end{aligned}\right\} \tag{474}$$

Substituting for ϕ, ψ, and $(A + B)$ according to equations (472), (473), and (460), we find that (474) is transformed to

$$\left.\begin{aligned} &\frac{1}{\sin q} \frac{\partial}{\partial q} \left\{ \frac{1}{\cos p + \cos q} \frac{\partial}{\partial \nu} (H_2 \sin^2 q) + \tfrac{1}{4}\psi_1 \sin q \right\} \\ &- \frac{1}{\sin p} \frac{\partial}{\partial p} \left\{ \frac{1}{\cos p + \cos q} \frac{\partial}{\partial \nu} (H_1 \sin^2 p) + \tfrac{1}{4}\phi_1 \sin p \right\} \\ &= \frac{2}{(\cos p + \cos q)^2} \frac{\partial}{\partial \nu} \{h_0 \cos p \cos q - h_1 \cos q + h_2 \cos p + h_5\} \,. \end{aligned}\right\} \tag{475}$$

On substituting for H_1 and H_2 according to equations (454) and (455), we find that equation (475) can be simplified to

$$\left.\begin{aligned} &2 \frac{\partial}{\partial \nu} (h_4 - h_3 - h_5) + \{2 + (\cos p + \cos q)^2\} \frac{dh_0}{d\nu} \\ &\quad + \frac{(\cos p + \cos q)^2}{4 \sin q} \frac{\partial}{\partial q} (\psi_1 \sin q) \\ &\quad - \frac{(\cos p + \cos q)^2}{4 \sin p} \frac{\partial}{\partial p} (\phi_1 \sin p) = 0 \,. \end{aligned}\right\} \tag{476}$$

Since ϕ_1 and ψ_1 are independent of q and p, respectively, we can write

$$\frac{1}{4 \sin p} \frac{\partial}{\partial p} (\phi_1 \sin p) = \Phi_1(p, \nu) \tag{477}$$

and

$$\frac{1}{4 \sin q} \frac{\partial}{\partial q} (\psi_1 \sin q) = \Psi_1(q, \nu) . \tag{478}$$

In terms of Φ_1 and Ψ_1 we can re-write equation (476) as

$$\left.\begin{aligned} 2 \frac{\partial}{\partial \nu} (h_4 - h_3 - h_5) + \{2 + (\cos p + \cos q)^2\} \frac{dh_0}{d\nu} \\ = (\cos p + \cos q)^2 \Phi_1 - (\cos p + \cos q)^2 \Psi_1 . \end{aligned}\right\} \tag{479}$$

On differentiating equation (479) partially with respect to p and q, we obtain, respectively,

$$\frac{\cos p + \cos q}{2 \sin p} \frac{\partial \Phi_1}{\partial p} = \Phi_1 - \Psi_1 - \frac{dh_0}{d\nu} , \tag{480}$$

$$\frac{\cos p + \cos q}{2 \sin q} \frac{\partial \Psi_1}{\partial q} = -\Phi_1 + \Psi_1 + \frac{dh_0}{d\nu} . \tag{481}$$

Adding equations (480) and (481), we obtain

$$\frac{1}{\sin p} \frac{\partial \Phi_1}{\partial p} + \frac{1}{\sin q} \frac{\partial \Psi_1}{\partial q} = 0 . \tag{482}$$

Since Φ_1 and Ψ_1 are independent of q and p, respectively, equation (482) implies that

$$\frac{1}{\sin p} \frac{\partial \Phi_1}{\partial p} = \alpha(\nu) ; \qquad \frac{1}{\sin q} \frac{\partial \Psi_1}{\partial q} = -\alpha(\nu) , \tag{483}$$

where $\alpha(\nu)$ is a function of ν only. Hence,

$$\left.\begin{aligned} \Phi_1 &= \tfrac{1}{4} g_1 - \alpha \cos p , \\ \Psi_1 &= \tfrac{1}{4} f_1 + \alpha \cos q , \end{aligned}\right\} \tag{484}$$

where f_1 and g_1 are functions of ν only. Substituting (484) in equation (480) or (481), we find

$$\tfrac{3}{2}(\cos p + \cos q)\alpha = \tfrac{1}{4}(g_1 - f_1) - \frac{dh_0}{d\nu} . \tag{485}$$

Hence, $\alpha = 0$, and consequently

$$g_1 - f_1 = 4\frac{dh_0}{d\nu} . \tag{486}$$

Hence, from (477) and (478) we now have

$$\frac{1}{\sin p}\frac{\partial}{\partial p}(\phi_1 \sin p) = g_1 ; \qquad \frac{1}{\sin q}\frac{\partial}{\partial q}(\psi_1 \sin q) = f_1 . \tag{487}$$

Equations (487) are easily integrated, and we find

$$\left.\begin{aligned} \phi_1 \sin p &= g_2 - g_1 \cos p , \\ \psi_1 \sin q &= f_2 - f_1 \cos q , \end{aligned}\right\} \tag{488}$$

where g_2 and f_2 are functions of ν only. Equations (476), (486), and (487) now give

$$\frac{\partial}{\partial \nu}(h_4 - h_3 - h_5 + h_0) = 0 , \tag{489}$$

or

$$h_4 - h_3 - h_5 + h_0 = \text{constant} . \tag{490}$$

We have now determined the dependence of the quantities A, B, F, G, and H on λ and μ. The only equations that remain to be considered are (iii), (viii), and (ix). These equations are seen to be equivalent to the following three equations:

$$\left.\begin{aligned} \frac{(\cos p + \cos q)^3}{4 \sin p \sin q}\frac{\partial C}{\partial p} &= -i(\cos p + \cos q)\frac{\partial \psi}{\partial \nu} \\ &\qquad - (A + B)\sin p - 4iH_2 \sin q , \end{aligned}\right\} \tag{491}$$

$$\left.\begin{aligned} \frac{(\cos p + \cos q)^3}{4 \sin p \sin q}\frac{\partial C}{\partial q} &= i(\cos p + \cos q)\frac{\partial \phi}{\partial \nu} \\ &\qquad - (A + B)\sin q + 4iH_1 \sin p , \end{aligned}\right\} \tag{492}$$

$$\tfrac{1}{2}(\cos p + \cos q)\frac{\partial C}{\partial \nu} = i\psi \sin q - i\phi \sin p . \tag{493}$$

On substituting for $(A + B)$, ϕ, and ψ according to equations (460), (472), (473), and (488) in (491) and (492), we find that these two equations are equivalent, respectively, to

$$\left.\begin{aligned}\frac{\partial C}{\partial p} = &-\frac{16i \sin p}{(\cos p + \cos q)^3} \frac{\partial^2}{\partial v^2}(H_2 \sin^2 q) - \frac{4i \sin p}{(\cos p + \cos q)^2}\left(\frac{df_2}{\partial v} - \frac{df_1}{\partial v}\cos q\right)\\ &- \frac{16i\,(h_0 \cos q + h_2)\sin p \cos p}{(\cos p + \cos q)^3} + \frac{16i \sin p}{(\cos p + \cos q)^3}(h_1 \cos q - h_5)\\ &- \frac{16i \sin p}{(\cos p + \cos q)^3} H_2 \sin^2 q\,,\end{aligned}\right\} \tag{494}$$

$$\left.\begin{aligned}\frac{\partial C}{\partial q} = &\frac{16i \sin q}{(\cos p + \cos q)^3} \frac{\partial^2}{\partial v^2}(H_1 \sin^2 p) + \frac{4i \sin q}{(\cos p + \cos q)^2}\left(\frac{dg_2}{dv} - \frac{dg_1}{dv}\cos p\right)\\ &- \frac{16i(h_0 \cos p - h_1)\sin q \cos q}{(\cos p + \cos q)^3} - \frac{16i \sin q}{(\cos p + \cos q)^3}(h_2 \cos p + h_5)\\ &+ \frac{16i \sin q}{(\cos p + \cos q)^3} H_1 \sin^2 p\,.\end{aligned}\right\} \tag{495}$$

Equations (494) and (495) should satisfy the integrability condition

$$\frac{\partial^2 C}{\partial q \partial p} = \frac{\partial^2 C}{\partial p \partial q}\,. \tag{496}$$

On evaluating the partial derivative with respect to q of the right-hand side of (494) and equating it to the partial derivative with respect to p of the left-hand side of (495), we find

$$\left.\begin{aligned}&12\frac{\partial^2}{\partial v^2}\{2(h_3 - h_1\cos p) - h_0 \sin^2 p\} + 8(\cos p + \cos q)\frac{\partial^2}{\partial v^2}(h_1 - h_0 \cos p)\\ &+ 2(\cos p + \cos q)\left(\frac{dg_2}{dv} - \frac{dg_1}{dv}\cos p\right) + (\cos p + \cos q)^2\frac{dg_1}{\partial v}\\ &- 12(h_0 \cos p - h_1)\cos q + 4h_0(\cos p + \cos q)\cos q - 12(h_2 \cos p + h_5)\\ &+ 4h_2(\cos p + \cos q) + 12\{2(h_3 - h_1 \cos p) - h_0 \sin^2 p\}\\ &+ 8(\cos p + \cos q)(h_1 - h_0 \cos p) + 12\frac{\partial^2}{\partial v^2}\{2(h_4 - h_2 \cos q) + h_0 \sin^2 q\}\\ &+ 8(\cos p + \cos q)\frac{\partial^2}{\partial v^2}(h_2 + h_0 \cos q) + 2(\cos p + \cos q)\left(\frac{df_2}{dv} - \frac{df_1}{dv}\cos q\right)\\ &+ (\cos p + \cos q)^2\frac{df_1}{dv} + 12(h_0 \cos q + h_2)\cos p - 4h_0(\cos p + \cos q)\cos p\\ &- 12(h_1 \cos q - h_5) + 4h_1(\cos p + \cos q) + 12\{2(h_4 - h_2 \cos q) + h_0 \sin^2 q\}\\ &+ 8(\cos p + \cos q)(h_2 + h_0 \cos q) = 0\,.\end{aligned}\right\} \tag{497}$$

On collecting together all the similar terms and using equation (486), we find that equation (497) can be simplified to the form

$$\left.\begin{aligned}&\left(-16\frac{d^2h_1}{dv^2}-12h_1+8\frac{d^2h_2}{dv^2}+12h_2+2\frac{dg_2}{dv}+2\frac{df_2}{dv}\right)\cos p\\&+\left(8\frac{d^2h_1}{dv^2}+12h_1-16\frac{d^2h_2}{dv^2}-12h_2+2\frac{dg_2}{dv}+2\frac{df_2}{dv}\right)\cos q\\&+24\frac{\partial^2}{\partial v^2}(h_3+h_4)+24(h_3+h_4)=0\,.\end{aligned}\right\}\tag{498}$$

Hence,

$$-4\frac{d^2h_1}{dv^2}-3h_1+2\frac{d^2h_2}{dv^2}+3h_2+\frac{1}{2}\frac{d}{dv}(f_2+g_2)=0\,,\tag{499}$$

$$+2\frac{d^2h_1}{dv^2}+3h_1-4\frac{d^2h_2}{dv^2}-3h_2+\frac{1}{2}\frac{d}{dv}(f_2+g_2)=0\,,\tag{500}$$

and

$$\frac{d^2}{dv^2}(h_3+h_4)=-(h_3+h_4)\,.\tag{501}$$

Adding equations (499) and (500), we obtain

$$2\frac{d^2}{dv^2}(h_1+h_2)=\frac{d}{dv}(f_2+g_2)\,.\tag{502}$$

Similarly, subtracting equation (499) from (500), we find

$$\frac{d^2}{dv^2}(h_1-h_2)=-(h_1-h_2)\,.\tag{503}$$

Hence,

$$h_3+h_4=\kappa_1\cos v+\kappa_2\sin v\,,\tag{504}$$

$$h_1-h_2=\kappa_3\cos v+\kappa_4\sin v\,,\tag{505}$$

$$\frac{d}{dv}(h_1+h_2)=\tfrac{1}{2}(f_2+g_2)-\kappa_5\,,\tag{506}$$

where $\kappa_1, \ldots, \kappa_5$ are constants.

On integrating equation (494), we obtain

$$\left.\begin{aligned} C = &-\frac{8i}{(\cos p+\cos q)^2}\frac{\partial^2}{\partial \nu^2}(H_2\sin^2 q)-\frac{4i}{\cos p+\cos q}\left(\frac{df_2}{d\nu}-\frac{df_1}{d\nu}\cos q\right)\\ &-\frac{8i(h_0\cos q+h_2)(2\cos p+\cos q)}{(\cos p+\cos q)^2}+\frac{8i}{(\cos p+\cos q)^2}(h_1\cos q-h_5)\\ &-\frac{8i}{(\cos p+\cos q)^2}H_2\sin^2 q+4i\gamma_1(q,\nu)\,, \end{aligned}\right\} \quad (507)$$

where $\gamma_1(q, \nu)$ is a function of q and ν only. Similarly, from equation (495) we derive

$$\left.\begin{aligned} C = &\frac{8i}{(\cos p+\cos q)^2}\frac{\partial^2}{\partial \nu^2}(H_1\sin^2 p)+\frac{4i}{(\cos p+\cos q)}\left(\frac{dg_2}{d\nu}-\frac{dg_1}{d\nu}\cos p\right)\\ &-\frac{8i(h_0\cos p-h_1)(2\cos q+\cos p)}{(\cos p+\cos q)^2}-\frac{8i}{(\cos p+\cos q)^2}(h_2\cos p+h_5)\\ &+\frac{8i}{(\cos p+\cos q)^2}H_1\sin^2 p+4i\gamma_2(p,\nu)\,, \end{aligned}\right\} \quad (508)$$

where γ_2 is independent of q. On equating equations (507), and (508), we find that both γ_1 and γ_2 can depend only on ν. We can therefore write

$$\gamma_2 \equiv \gamma(\nu)\,. \quad (509)$$

We have finally to consider equation (493). On substituting for C, ϕ, and ψ according to equations (508), (472), (473), and (488) in (493), we find that this equation becomes

$$\left.\begin{aligned} &4\frac{\partial^3}{\partial\nu^3}\{2(h_3-h_1\cos p)-h_0\sin^2 p\}+2(\cos p+\cos q)\left(\frac{d^2g_2}{d\nu^2}-\frac{d^2g_1}{d\nu^2}\cos p\right)\\ &-4(2\cos q+\cos p)\frac{\partial}{\partial\nu}(h_0\cos p-h_1)-4\frac{\partial}{\partial\nu}(h_2\cos p+h_5)\\ &+4\frac{\partial}{\partial\nu}\{2(h_3-h_1\cos p)-h_0\sin^2 p\}+2(\cos p+\cos q)^2\frac{d\gamma}{d\nu}\\ &+4\frac{\partial}{\partial\nu}\{2(h_3-h_1\cos p)-h_0\sin^2 p\}+(\cos p+\cos q)g_2\\ &-g_1(\cos p+\cos q)\cos p-4\frac{\partial}{\partial\nu}\{2(h_4-h_2\cos q)+h_0\sin^2 q\}\\ &-(\cos p+\cos q)f_2+f_1(\cos p+\cos q)\cos q=0\,. \end{aligned}\right\} \quad (510)$$

The terms which occur as coefficients of cos p and cos q in the foregoing equation yield, respectively,

$$-8\frac{d^3h_1}{dv^3} + 2\frac{d^2g_2}{dv^2} - 12\frac{dh_1}{dv} - 4\frac{dh_2}{dv} + g_2 - f_2 = 0 \qquad (511)$$

and

$$2\frac{d^2g_2}{dv^2} + 8\frac{dh_1}{dv} + 8\frac{dh_2}{dv} + g_2 - f_2 = 0\,. \qquad (512)$$

On subtracting equation (512) from (511), we find that

$$-8\frac{d^3h_1}{dv^3} - 20\frac{dh_1}{dv} - 12\frac{dh_2}{dv} = 0\,, \qquad (513)$$

or, after integrating,

$$2\frac{d^2h_1}{dv^2} + 5h_1 + 3h_2 = 4\kappa_6\,, \qquad (514)$$

where κ_6 is a constant. On the other hand, according to equation (503),

$$2\frac{d^2}{dv^2}(h_1 - h_2) + 2h_1 - 2h_2 = 0\,. \qquad (515)$$

Subtracting equation (515) from (514), we obtain

$$2\frac{d^2h_2}{dv^2} + 3h_1 + 5h_2 = 4\kappa_6\,. \qquad (516)$$

And now, adding (514) and (516), we have

$$2\frac{d^2}{dv^2}(h_1 + h_2) + 8(h_1 + h_2) = 8\kappa_6\,, \qquad (517)$$

or

$$\frac{d^2}{dv^2}(h_1 + h_2) = -4(h_1 + h_2) + 4\kappa_6\,. \qquad (518)$$

The solution of (518) is readily seen to be

$$h_1 + h_2 = \kappa_7 \cos 2\nu + \kappa_8 \sin 2\nu + \kappa_6 , \tag{519}$$

where κ_7 and κ_8 are constants. From equation (506) we have

$$f_2 + g_2 = 2 \frac{d}{d\nu} (h_1 + h_2) + 2\kappa_5 , \tag{520}$$

or, according to (519),

$$f_2 + g_2 = 4(-\kappa_7 \sin 2\nu + \kappa_8 \cos 2\nu) + 2\kappa_5 . \tag{521}$$

Using equation (520), equation (512) can be reduced to

$$2 \frac{d^2 g_2}{d\nu^2} + 5g_2 + 3f_2 = 8\kappa_5 . \tag{522}$$

Using (521) to eliminate f_2 in (522), we find

$$\frac{d^2 g_2}{d\nu^2} + g_2 = -6(-\kappa_7 \sin 2\nu + \kappa_8 \cos 2\nu) + \kappa_5 . \tag{523}$$

The solution of (523) is readily seen to be

$$g_2 = (\kappa_9 \cos \nu + \kappa_{10} \sin \nu) + 2(-\kappa_7 \sin 2\nu + \kappa_8 \cos 2\nu) + \kappa_5 . \tag{524}$$

From (521) we find

$$f_2 = -(\kappa_9 \cos \nu + \kappa_{10} \sin \nu) + 2(-\kappa_7 \sin 2\nu + \kappa_8 \cos 2\nu) + \kappa_5 . \tag{525}$$

In equations (524) and (525), κ_9 and κ_{10} are again constants.

On collecting all the terms in equation (510) which do not occur as the coefficients of either $\cos p$ or $\cos q$, we find

$$\left.\begin{aligned} & 8\frac{d^3h_3}{d\nu^3} - 4\frac{dh_5}{d\nu} + 16\frac{dh_3}{d\nu} - 8\frac{dh_4}{d\nu} - 2(\cos p + \cos q)\cos p\,\frac{d^2g_1}{d\nu^2} \\ & \quad - 4\sin^2 p\,\frac{d^3h_0}{d\nu^3} + \{-4(2\cos q + \cos p)\cos p - 8\sin^2 p - 4\sin^2 q\}\frac{dh_0}{d\nu} \\ & \quad + 2(\cos p + \cos q)^2\frac{d\gamma}{d\nu} - g_1(\cos p + \cos q)\cos p \\ & \quad + f_1(\cos p + \cos q)\cos q = 0 . \end{aligned}\right\} \tag{526}$$

On using equation (486) and after rearranging the different terms, we find that equation (526) can be reduced to

$$\left.\begin{aligned} & 8\frac{d^3h_3}{dv^3} - 4\frac{dh_5}{dv} + 16\frac{dh_3}{dv} - 8\frac{dh_4}{dv} + (4\cos p\cos q + 2 + \cos 2p + \cos 2q)\frac{d\gamma}{dv} \\ & - (2\tfrac{1}{2} - \tfrac{1}{2}\cos 2q + 3\cos p\cos q)g_1 + (2\tfrac{1}{2} - \tfrac{1}{2}\cos 2p + 3\cos p\cos q)f_1 \\ & - (1\tfrac{1}{2} + \tfrac{1}{2}\cos 2p + 2\cos p\cos q)\frac{d^2g_1}{dv^2} + \tfrac{1}{2}(1 - \cos 2p)\frac{d^2f_1}{dv^2} = 0\,. \end{aligned}\right\} \quad (527)$$

On equating separately to zero the terms which occur as the coefficients of $\cos p\cos q$, $\cos 2p$, and $\cos 2q$ in (527), we find, respectively,

$$4\frac{d\gamma}{dv} - 3g_1 + 3f_1 - 2\frac{d^2g_1}{dv^2} = 0\,, \quad (528)$$

$$\frac{d\gamma}{dv} - \tfrac{1}{2}f_1 - \frac{1}{2}\frac{d^2g_1}{dv^2} - \frac{1}{2}\frac{d^2f_1}{dv^2} = 0\,, \quad (529)$$

$$\frac{d\gamma}{dv} + \tfrac{1}{2}g_1 = 0\,. \quad (530)$$

The remaining terms in (527) give

$$\left.\begin{aligned} 8\frac{d^3h_3}{dv^3} - 4\frac{dh_5}{dv} + 16\frac{dh_3}{dv} - 8\frac{dh_4}{dv} + 2\frac{d\gamma}{dv} - \tfrac{5}{2}(g_1 - f_1) & \\ - \frac{3}{2}\frac{d^2g_1}{dv^2} + \frac{1}{2}\frac{d^2f_1}{dv^2} = 0\,. & \end{aligned}\right\} \quad (531)$$

From equations (529) and (530) we easily find that

$$\frac{d^2}{dv^2}(f_1 + g_1) + (f_1 + g_1) = 0\,. \quad (532)$$

Similarly, from equations (528) and (530) we obtain

$$2\frac{d^2g_1}{dv^2} + 5g_1 - 3f_1 = 0\,. \quad (533)$$

From equations (532) and (533) we readily obtain

$$\frac{d^2}{dv^2}(f_1 - g_1) + 4(f_1 - g_1) = 0\,. \quad (534)$$

Hence,

$$f_1 + g_1 = \kappa_{11} \cos \nu + \kappa_{12} \sin \nu , \tag{535}$$

$$f_1 - g_1 = \kappa_{13} \cos 2\nu + \kappa_{14} \sin 2\nu . \tag{536}$$

From (486) we have

$$4 \frac{dh_0}{d\nu} = -\kappa_{13} \cos 2\nu - \kappa_{14} \sin 2\nu , \tag{537}$$

or

$$4h_0 = -\tfrac{1}{2}(\kappa_{13} \sin 2\nu - \kappa_{14} \cos 2\nu) + \kappa_{15} . \tag{538}$$

Again, from equation (530)

$$4 \frac{d\gamma}{d\nu} = -2g_1 = \kappa_{13} \cos 2\nu + \kappa_{14} \sin 2\nu - \kappa_{11} \cos \nu - \kappa_{12} \sin \nu , \tag{539}$$

or

$$4\gamma = \tfrac{1}{2}(\kappa_{13} \sin 2\nu - \kappa_{14} \cos 2\nu) - \kappa_{11} \sin \nu + \kappa_{12} \cos \nu + \kappa_{16} . \tag{540}$$

After some minor transformations equation (531) can be transformed to

$$8 \frac{d^3h_3}{d\nu^3} - 4 \frac{dh_5}{d\nu} + 16 \frac{dh_3}{d\nu} - 8 \frac{dh_4}{d\nu} + g_1 - f_1 = 0 . \tag{541}$$

From (536) and (541) we now have

$$8 \frac{d^3h_3}{d\nu^3} - 4 \frac{dh_5}{d\nu} + 16 \frac{dh_3}{d\nu} - 8 \frac{dh_4}{d\nu} = \kappa_{13} \cos 2\nu + \kappa_{14} \sin 2\nu , \tag{542}$$

or, after integration,

$$\left. \begin{aligned} & 8 \frac{d^2h_3}{d\nu^2} - 4h_5 + 16h_3 - 8h_4 \\ & \qquad = \tfrac{1}{2}(\kappa_{13} \sin 2\nu - \kappa_{14} \cos 2\nu) + \text{constant} . \end{aligned} \right\} \tag{543}$$

On the other hand, according to equation (490), we can write

$$h_4 - h_3 - h_5 + h_0 = \text{constant} = -\kappa_{17} . \tag{544}$$

Using equations (538) and (544), equation (543) is seen to reduce to

$$2\,\frac{d^2h_3}{dv^2} + 5h_3 - 3h_4 = 4\kappa_{18}\,. \tag{545}$$

But, according to equation (501),

$$2\,\frac{d^2}{dv^2}(h_3 + h_4) + 2h_3 + 2h_4 = 0\,. \tag{546}$$

From equations (545) and (546) we readily obtain

$$\frac{d^2}{dv^2}(h_3 - h_4) + 4(h_3 - h_4) = 4\kappa_{18}\,. \tag{547}$$

The solution of (547) is seen to be

$$h_3 - h_4 = \kappa_{19}\cos 2v + \kappa_{20}\sin 2v + \kappa_{18}\,. \tag{548}$$

We have now determined all the quantities in terms of p, q, and v. Collecting our results, we have

$$\left.\begin{aligned}
A - B &= -2i(H_1 - H_2)\,,\\
A + B &= \frac{4i}{\sin p\sin q}(h_0\cos p\cos q - h_1\cos q\\
&\qquad\qquad + h_2\cos p + h_5)\,,\\
C &= \frac{8i}{(\cos p + \cos q)^2}\left\{\frac{\partial^2}{\partial v^2}(H_1\sin^2 p)\right.\\
&\quad - (2\cos q + \cos p)(h_0\cos p - h_1) - (h_2\cos p + h_5)\\
&\quad \left. + H_1\sin^2 p\right\} + \frac{4i}{(\cos p + \cos q)}\left(\frac{dg_2}{dv} - \frac{dg_1}{dv}\cos p\right) + 4i\gamma\,,\\
H &= H_1 + H_2\,,\\
\phi &= iF - G = \frac{4\sin p}{\cos p + \cos q}\,\frac{\partial H_1}{\partial v} + \frac{1}{\sin p}(g_2 - g_1\cos p)\,,\\
\psi &= iF + G = \frac{4\sin q}{\cos p + \cos q}\,\frac{\partial H_2}{\partial v} + \frac{1}{\sin q}(f_2 - f_1\cos q)\,,
\end{aligned}\right\} \tag{549}$$

where

$$p = i\lambda + \mu\,; \qquad q = i\lambda - \mu \tag{550}$$

and

$$\left.\begin{aligned} H_1 &= \frac{2}{\sin^2 p}(h_3 - h_1 \cos p) - h_0 \,, \\ H_2 &= \frac{2}{\sin^2 q}(h_4 - h_2 \cos q) + h_0 \,. \end{aligned}\right\} \qquad (551)$$

In the foregoing solution, h_0, h_1, h_2, h_3, h_4, h_5, f_1, f_2, g_1, g_2, and γ are all functions of ν only and are given by

$$\left.\begin{aligned} 4h_0 &= -\tfrac{1}{2}(\kappa_{13} \sin 2\nu - \kappa_{14} \cos 2\nu) + \kappa_{15} \,, \\ h_1 &= \tfrac{1}{2}(\kappa_3 \cos \nu + \kappa_4 \sin \nu) + \tfrac{1}{2}(\kappa_7 \cos 2\nu + \kappa_8 \sin 2\nu) + \tfrac{1}{2}\kappa_6 \,, \\ h_2 &= -\tfrac{1}{2}(\kappa_3 \cos \nu + \kappa_4 \sin \nu) + \tfrac{1}{2}(\kappa_7 \cos 2\nu + \kappa_8 \sin 2\nu) + \tfrac{1}{2}\kappa_6 , \\ h_3 &= \tfrac{1}{2}(\kappa_1 \cos \nu + \kappa_2 \sin \nu) + \tfrac{1}{2}(\kappa_{19} \cos 2\nu + \kappa_{20} \sin 2\nu) + \tfrac{1}{2}\kappa_{18} , \\ h_4 &= \tfrac{1}{2}(\kappa_1 \cos \nu + \kappa_2 \sin \nu) - \tfrac{1}{2}(\kappa_{19} \cos 2\nu + \kappa_{20} \sin 2\nu) - \tfrac{1}{2}\kappa_{18} \,, \\ h_5 &= -(\kappa_{19} \cos 2\nu + \kappa_{20} \sin 2\nu) - \tfrac{1}{8}(\kappa_{13} \sin 2\nu - \kappa_{14} \cos 2\nu) + \kappa_{17} \,, \\ f_1 &= \tfrac{1}{2}(\kappa_{11} \cos \nu + \kappa_{12} \sin \nu) + \tfrac{1}{2}(\kappa_{13} \cos 2\nu + \kappa_{14} \sin 2\nu) \,, \\ g_1 &= \tfrac{1}{2}(\kappa_{11} \cos \nu + \kappa_{12} \sin \nu) - \tfrac{1}{2}(\kappa_{13} \cos 2\nu + \kappa_{14} \sin 2\nu) \,, \\ f_2 &= -(\kappa_9 \cos \nu + \kappa_{10} \sin \nu) + 2(-\kappa_7 \sin 2\nu + \kappa_8 \cos 2\nu) + \kappa_5 \,, \\ g_2 &= (\kappa_9 \cos \nu + \kappa_{10} \sin \nu) + 2(-\kappa_7 \sin 2\nu + \kappa_8 \cos 2\nu) + \kappa_5 \,, \\ 4\gamma &= \tfrac{1}{2}(\kappa_{13} \sin 2\nu - \kappa_{14} \cos 2\nu) - \kappa_{11} \sin \nu + \kappa_{12} \cos \nu + \kappa_{16} \,. \end{aligned}\right\} \qquad (552)$$

It will be recalled that the quantities A, B, C, F, G, and H are related to the coefficients of the velocity ellipsoid, a, b, c, f, g, and h according to the relations (cf. Eqs. [47], [426], and [427])

$$\left.\begin{aligned} a &= A(\cosh^2 \lambda - \cos^2 \mu) = -A \sin p \sin q \,, \\ b &= B(\cosh^2 \lambda - \cos^2 \mu) = -B \sin p \sin q \,, \\ c &= C \cosh^2 \lambda \cos^2 \mu \qquad = \tfrac{1}{4}C(\cos p + \cos q)^2 \,, \\ f &= F \cosh \lambda \cos \mu \sqrt{\cosh^2 \lambda - \cos^2 \mu} \\ &\qquad = \tfrac{1}{2}F(\cos p + \cos q)\sqrt{-\sin p \sin q} \,, \\ g &= G \cosh \lambda \cos \mu \sqrt{\cosh^2 \lambda - \cos^2 \mu} \\ &\qquad = \tfrac{1}{2}G(\cos p + \cos q)\sqrt{-\sin p \sin q} \,, \\ h &= H(\cosh^2 \lambda - \cos^2 \mu) = -H \sin p \sin q \,. \end{aligned}\right\} \qquad (553)$$

We thus see that the general solutions for a, b, c, f, g, and h involve twenty constants of integration, as before.

22. *The solution for the motions of the local centroids.*—We shall now consider the six equations (II_6) for ξ, η, and ζ. These quantities are related to the Δ's according to equation (48) of Part I.

From equations (i) and (ii) it follows that

$$\frac{\partial \xi}{\partial \lambda} = \frac{\partial \eta}{\partial \mu} . \tag{554}$$

But, according to equation (iv),

$$\frac{\partial \xi}{\partial \mu} = - \frac{\partial \eta}{\partial \lambda} . \tag{555}$$

From equations (554) and (555) it follows that

$$\frac{\partial^2 \xi}{\partial \lambda^2} + \frac{\partial^2 \xi}{\partial \mu^2} = 0 ; \qquad \frac{\partial^2 \eta}{\partial \lambda^2} + \frac{\partial^2 \eta}{\partial \mu^2} = 0 . \tag{556}$$

We should therefore have

$$\xi = \xi_1(i\lambda + \mu, \nu) + \xi_2(i\lambda - \mu, \nu) . \tag{557}$$

From equations (554) and (555) we now derive

$$\eta = i\xi_1(i\lambda + \mu, \nu) - i\xi_2(i\lambda + \mu, \nu) . \tag{558}$$

As in § 21, we introduce the variables p and q defined by

$$p = i\lambda + \mu ; \qquad q = i\lambda - \mu . \tag{559}$$

Instead of (557) and (558) we can write

$$\xi = \xi_1(p, \nu) + \xi_2(q, \nu) , \tag{560}$$

$$\eta = i\{\xi_1(p, \nu) - \xi_2(q, \nu)\} . \tag{561}$$

Using equations (445), (560), and (561), equation (i) can be reduced to the form

$$\left(\sin p \frac{\partial \xi_1}{\partial p} + \xi_1 \cos p\right) \sin q + \left(\sin q \frac{\partial \xi_2}{\partial q} + \xi_2 \cos q\right) \sin p = 0 . \quad (562)$$

Since ξ_1 and ξ_2 are independent of q and p, respectively, (562) implies that

$$\sin p \frac{\partial \xi_1}{\partial p} + \xi_1 \cos p = \alpha_0 \sin p , \quad (563)$$

$$\sin q \frac{\partial \xi_2}{\partial q} + \xi_2 \cos q = -\alpha_0 \sin q , \quad (564)$$

where α_0 is a function of ν only. Equations (563) and (564) are linear equations in ξ_1 and ξ_2, respectively. They are readily integrated, and it is found that

$$\xi_1 = \frac{1}{\sin p} (\alpha_1 - \alpha_0 \cos p) , \quad (565)$$

$$\xi_2 = \frac{1}{\sin q} (\alpha_2 + \alpha_0 \cos q) , \quad (566)$$

where α_1 and α_2 are again functions of ν only.

From equations (v), (vi), and (iii) we can obtain the following three equations:

$$\tfrac{1}{4}(\cos p + \cos q)^2 \frac{\partial \zeta}{\partial p} = -i \sin p \sin q \frac{\partial \xi_2}{\partial \nu} , \quad (567)$$

$$\tfrac{1}{4}(\cos p + \cos q)^2 \frac{\partial \zeta}{\partial q} = -i \sin p \sin q \frac{\partial \xi_1}{\partial \nu} , \quad (568)$$

and

$$(\cos p + \cos q) \frac{\partial \zeta}{\partial \nu} = 2i(\xi_1 \sin p + \xi_2 \cos q) . \quad (569)$$

Using the solutions for ξ_1 and ξ_2 given by (565) and (566), the foregoing three equations take the forms

$$(\cos p + \cos q)^2 \frac{\partial\zeta}{\partial p} = -4i\left(\frac{d\alpha_2}{dv} + \frac{d\alpha_0}{dv}\cos q\right)\sin p\,, \tag{570}$$

$$(\cos p + \cos q)^2 \frac{d\zeta}{dq} = -4i\left(\frac{d\alpha_1}{dv} - \frac{d\alpha_0}{dv}\cos p\right)\sin q\,, \tag{571}$$

and

$$\left.\begin{aligned}\frac{\partial\zeta}{\partial v} = \frac{2i}{\cos p + \cos q}\{(\alpha_1 + \alpha_2) \\ - \alpha_0(\cos p - \cos q)\}\,.\end{aligned}\right\} \tag{572}$$

From equation (570) we obtain

$$(\cos p + \cos q)^2 \frac{\partial^2\zeta}{\partial v\partial p} = -\,4i\left(\frac{d^2\alpha_2}{dv^2} + \frac{d^2\alpha_0}{dv^2}\cos q\right)\sin p\,. \tag{573}$$

On the other hand, from (572) we obtain

$$\left.\begin{aligned}(\cos p + \cos q)^2 \frac{\partial^2\zeta}{\partial p\partial v} = 2i\{(\alpha_1 + \alpha_2) - \alpha_0(\cos p - \cos q)\}\sin p \\ + 2i\alpha_0(\cos p + \cos q)\sin p\,.\end{aligned}\right\} \tag{574}$$

Equating the left-hand sides of the equations (573) and (574), we obtain

$$\alpha_1 + \alpha_2 + 2\alpha_0\cos q = -\,2\frac{d^2\alpha_2}{dv^2} - 2\frac{d^2\alpha_0}{dv^2}\cos q\,. \tag{575}$$

Similarly, from equations (571) and (572) we obtain

$$\alpha_1 + \alpha_2 - 2\alpha_0\cos p = -\,2\frac{d^2\alpha_1}{dv^2} + 2\frac{d^2\alpha_0}{dv^2}\cos p\,. \tag{576}$$

From equations (575) and (576) we derive

$$2\frac{d^2\alpha_2}{dv^2} = -\,(\alpha_1 + \alpha_2)\,, \tag{577}$$

$$2\frac{d^2\alpha_1}{dv^2} = -\,(\alpha_1 + \alpha_2)\,, \tag{578}$$

and

$$\frac{d^2\alpha_0}{d\nu^2} = -\alpha_0 . \tag{579}$$

Equations (577) and (578) can be combined to give

$$\frac{d^2}{d\nu^2}(\alpha_1 + \alpha_2) + (\alpha_1 + \alpha_2) = 0 ; \qquad \frac{d^2}{d\nu^2}(\alpha_1 - \alpha_2) = 0 . \tag{580}$$

Solving the equations (579) and (580), we have

$$\alpha_1 + \alpha_2 = 2(k_1 \cos\nu + k_2 \sin\nu) , \tag{581}$$

$$\alpha_1 - \alpha_2 = 2(k_3 + k^*\nu) , \tag{582}$$

$$\alpha_0 = k_4 \cos\nu + k_5 \sin\nu , \tag{583}$$

where k_1, k_2, k_3, k_4, k_5, and k^* are constants.

Substituting for $(\alpha_1 + \alpha_2)$ from (581) in equation (572), we obtain

$$\frac{\partial\zeta}{\partial\nu} = \frac{2i}{\cos p + \cos q}\{2(k_1 \cos\nu + k_2 \sin\nu) - \alpha_0(\cos p - \cos q)\} . \tag{584}$$

Integrating the foregoing equation, we have

$$\left.\zeta = \frac{2i}{(\cos p + \cos q)}\left\{2(k_1 \sin\nu - k_2 \cos\nu) + (\cos p - \cos q)\frac{d\alpha_0}{d\nu}\right\} + \zeta_1(p, q) ,\right\} \tag{585}$$

where ζ_1 is independent of ν. Substituting for ζ according to the foregoing equation in (570) and (571), we obtain, after some simplifications,

$$\frac{\partial\zeta_1}{\partial p} = \frac{4ik^* \sin p}{(\cos p + \cos q)^2} ; \qquad \frac{\partial\zeta_1}{\partial q} = -\frac{4ik^* \sin q}{(\cos p + \cos q)^2} . \tag{586}$$

These two equations are seen to be incompatible if $k^* \neq 0$. Hence,

$$k^* = 0 ; \qquad \frac{\partial\zeta_1}{\partial p} = \frac{\partial\zeta_1}{\partial q} = 0 ; \tag{587}$$

in other words, ζ_1 is a constant ($=2k_6$, say). With this we have determined all the quantities in terms of p, q, and ν.

Collecting our results, we have

$$\xi = \xi_1 + \xi_2; \qquad \eta = i(\xi_1 - \xi_2)\,, \tag{588}$$

where

$$\xi_1 = \frac{1}{\sin p}(\alpha_1 - \alpha_0 \cos p)\;; \qquad \xi_2 = \frac{1}{\sin q}(\alpha_2 + \alpha_0 \cos q)\,. \tag{589}$$

In the foregoing equations α_1, α_2, and α_0 are functions of ν only and are given by

$$\left.\begin{aligned} \alpha_1 &= k_1 \cos\nu + k_2 \sin\nu + k_3\,,\\ \alpha_2 &= k_1 \cos\nu + k_2 \sin\nu - k_3\,,\\ \alpha_0 &= k_4 \cos\nu + k_5 \sin\nu\,. \end{aligned}\right\} \tag{590}$$

Finally,

$$\zeta = \frac{2i}{\cos p + \cos q}\left\{ 2(k_1 \sin\nu - k_2 \cos\nu) + (\cos p - \cos q)\frac{d\alpha_0}{d\nu}\right\} + 2k_6\,. \tag{591}$$

In terms of the variables λ and μ the expressions for ξ, η, and ζ are found to be

$$\left.\begin{aligned} \xi &= \frac{2k_3 \cosh\lambda \sin\mu - 2i(k_1 \cos\nu + k_2 \sin\nu)\sinh\lambda \cos\mu - (k_4 \cos\nu + k_5 \sin\nu)\sin 2\mu}{\cosh^2\lambda - \cos^2\mu}\,,\\ \eta &= \frac{2k_3 \sinh\lambda \cos\mu + 2i(k_1 \cos\nu + k_2 \sin\nu)\cosh\lambda \sin\mu - (k_4 \cos\nu + k_5 \sin\nu)\sinh 2\lambda}{\cosh^2\lambda - \cos^2\mu}\,,\\ \zeta &= \frac{2i(k_1 \sin\nu - k_2 \cos\nu) + 2\,(-k_4 \sin\nu + k_5 \cos\nu)\sinh\lambda \sin\mu}{\cosh\lambda \cos\mu} + 2k_6\,. \end{aligned}\right\} \tag{592}$$

It may be recalled that, according to equations (48), (426), and (427), ξ, η, and ζ are related to the Δ's by

$$\left.\begin{aligned} \Delta_1 &= \xi\sqrt{\cosh^2\lambda - \cos^2\mu} = \xi\sqrt{-\sin p \sin q}\,,\\ \Delta_2 &= \eta\sqrt{\cosh^2\lambda - \cos^2\mu} = \eta\sqrt{-\sin p \sin q}\,,\\ \Delta_3 &= \zeta \cosh\lambda \cos\mu \qquad\quad = \tfrac{1}{2}\zeta(\cos p + \cos q)\,. \end{aligned}\right\} \tag{593}$$

23. *The discussion of the potential.*—We have already proved in Part IV that for stellar systems with differential motions the potential function $\mathfrak{V}$ must necessarily be characterized by an axis of helical symmetry. We shall now show that if the axis of helical symmetry be chosen as the z-axis of the fundamental frame of reference, then in the solutions for ξ, η, and ζ obtained in § 22, k_1, k_2, k_4, and k_5 are all zero. The proof is as follows:

According to (III_6) we have

$$\xi \frac{\partial \mathfrak{V}}{\partial \lambda} + \eta \frac{\partial \mathfrak{V}}{\partial \mu} + \zeta \frac{\partial \mathfrak{V}}{\partial \nu} = 0 . \tag{594}$$

Using equation (588) of § 22, we can re-write (594) as

$$\xi_1 \left(\frac{\partial \mathfrak{V}}{\partial \lambda} + i \frac{\partial \mathfrak{V}}{\partial \mu} \right) + \xi_2 \left(\frac{\partial \mathfrak{V}}{\partial \lambda} - i \frac{\partial \mathfrak{V}}{\partial \mu} \right) + \zeta \frac{\partial \mathfrak{V}}{\partial \nu} = 0 ; \tag{595}$$

or, according to (445′), equation (595) is equivalent to

$$2i\xi_1 \frac{\partial \mathfrak{V}}{\partial p} + 2i\xi_2 \frac{\partial \mathfrak{V}}{\partial q} + \zeta \frac{\partial \mathfrak{V}}{\partial \nu} = 0 . \tag{596}$$

If we put $k_1 = k_2 = k_4 = k_5 = 0$ in equations (589), (590), and (591), we obtain

$$\xi_1 = \frac{k_3}{\sin p} ; \qquad \xi_2 = -\frac{k_3}{\sin q} ; \qquad \zeta = 2k_6 . \tag{597}$$

Equation (596) now becomes

$$\frac{k_3}{\sin p} \frac{\partial \mathfrak{V}}{\partial p} - \frac{k_3}{\sin q} \frac{\partial \mathfrak{V}}{\partial q} - ik_6 \frac{\partial \mathfrak{V}}{\partial \nu} = 0 . \tag{598}$$

The Lagrangian subsidiary equations of (598) are

$$\frac{\sin p\,dp}{k_3} = -\frac{\sin q\,dq}{k_3} = i \frac{d\nu}{k_6} . \tag{599}$$

From equation (599) it readily follows that

$$\left.\begin{aligned} \cos p + i\,\frac{k_3}{k_6}\,\nu &= \text{constant} \\ \text{and} \qquad \cos q - i\,\frac{k_3}{k_6}\,\nu &= \text{constant} \end{aligned}\right\} \tag{600}$$

are two first integrals. An alternative form for the two first integrals are

$$\left.\begin{aligned} \cos p + \cos q &= \text{constant}\,, \\ (\cos p - \cos q) + 2i\,\frac{k_3}{k_6}\,\nu &= \text{constant}\,. \end{aligned}\right\} \tag{601}$$

But

$$\left.\begin{aligned} \cos p + \cos q &= 2\cosh\lambda\cos\mu\,, \\ \cos p - \cos q &= -2i\sinh\lambda\sin\mu\,. \end{aligned}\right\} \tag{602}$$

Thus, the two integrals (601) are equivalent to

$$\left.\begin{aligned} \cosh\lambda\cos\mu &= \text{constant}\,, \\ \sinh\lambda\sin\mu - \frac{k_3}{k_6}\,\nu &= \text{constant}\,. \end{aligned}\right\} \tag{603}$$

Hence,

$$\mathfrak{B}(\lambda, \mu, \nu) \equiv \mathfrak{B}\left(\cosh\lambda\cos\mu, \sinh\lambda\sin\mu - \frac{k_3}{k_6}\,\nu\right). \tag{604}$$

On the other hand, according to equation (423),

$$\left.\begin{aligned} (x^2 + y^2)^{1/2} = \cosh\lambda\cos\mu\,; \qquad z = \sinh\lambda\sin\mu\,; \\ \nu = \tan^{-1}\frac{y}{x}\,. \end{aligned}\right\} \tag{605}$$

Equation (604) is therefore equivalent to

$$\mathfrak{B}(x, y, z) \equiv \mathfrak{B}\left(x^2 + y^2;\, z - \frac{k_3}{k_6}\tan^{-1}\frac{y}{x}\right). \tag{606}$$

In other words, the potential function has a helical symmetry about the z-axis. This proves the statement made at the beginning of this section.

Thus, if we choose the z-axis of the fundamental frame of reference to coincide with the axis of helical symmetry of $\mathfrak{V}$, our expressions for ξ, η, and ζ are considerably simplified. They are obtained by putting $k_1 = k_2 = k_4 = k_5 = 0$ in (592):

$$\left.\begin{aligned} \xi &= \frac{2k_3 \cosh \lambda \sin \mu}{\cosh^2 \lambda - \cos^2 \mu}\,, \\ \eta &= \frac{2k_3 \sinh \lambda \cos \mu}{\cosh^2 \lambda - \cos^2 \mu}\,, \\ \zeta &= 2k_6\,. \end{aligned}\right\} \tag{607}$$

The corresponding solutions for the Δ's are (cf. Eq. [593])

$$\left.\begin{aligned} \Delta_1 &= \frac{2k_3 \cosh \lambda \sin \mu}{\sqrt{\cosh^2 \lambda - \cos^2 \mu}}\,, \\ \Delta_2 &= \frac{2k_3 \sinh \lambda \cos \mu}{\sqrt{\cosh^2 \lambda - \cos^2 \mu}}\,, \\ \Delta_3 &= 2k_6 \cosh \lambda \cos \mu\,. \end{aligned}\right\} \tag{608}$$

From equation (606) it follows that for stellar systems for which the z-axis is an axis of symmetry, k_3 must also vanish, for then

$$\mathfrak{V}(x, y, z) \equiv \mathfrak{V}(x^2 + y^2, z)\,. \tag{609}$$

For this case, according to equation (608),

$$\Delta_1 = 0; \qquad \Delta_2 = 0; \qquad \Delta_3 = 2k_6 \cosh \lambda \cos \mu. \tag{610}$$

24. *A special case.*—We shall illustrate the use of the general solutions obtained in §§ 21 and 22 by considering a special case. Let us assume that the principal axes of the velocity ellipsoid are along the principal directions at the point considered. If this is the case, then

$$F = G = H = 0\,. \tag{611}$$

From the soution obtained in § 21 (cf. Eqs. [549]–[551]) we readily verify that (611) implies that

$$\frac{dh_0}{d\nu} = h_1 = h_2 = h_3 = h_4 = g_1 = f_1 = g_2 = f_2 = 0 . \qquad (612)$$

From (552) we now see that, according to (612), all the constants of integration except κ_{15}, κ_{16}, and κ_{17} are zero. Hence, for the case under consideration

$$H_1 = -H_2 = -h_0 = -\tfrac{1}{4}\kappa_{15} ; \quad h_5 = \kappa_{17} ; \quad \gamma = \tfrac{1}{4}\kappa_{16} \qquad (613)$$

and

$$\left.\begin{aligned} A - B &= i\kappa_{15} , \\ A + B &= \frac{4i}{\sin p \sin q} (\tfrac{1}{4}\kappa_{15} \cos p \cos q + \kappa_{17}) , \\ C &= \frac{8i}{(\cos p + \cos q)^2} (-\tfrac{1}{2}\kappa_{15} \cos p \cos q - \tfrac{1}{4}\kappa_{15} - \kappa_{17}) \\ &\qquad + i\kappa_{16} . \end{aligned}\right\} \qquad (614)$$

Equations (614) can be written alternatively in the forms

$$\left.\begin{aligned} -(A - B)\sin p \sin q &= -i\kappa_{15} \sin p \sin q , \\ -(A + B)\sin p \sin q &= -i\kappa_{15} \cos p \cos q - 4i\kappa_{17} , \\ \tfrac{1}{4}C(\cos p + \cos q)^2 &= -i\kappa_{15} \cos p \cos q - \tfrac{1}{2}i\kappa_{15} - 2i\kappa_{17} \\ &\qquad + \tfrac{1}{4}i\kappa_{16}(\cos p + \cos q)^2 . \end{aligned}\right\} \qquad (615)$$

In terms of the variables λ and μ we find that equations (615) are equivalent to

$$\left.\begin{aligned} (A - B)(\cosh^2 \lambda - \cos^2 \mu) &= a_1(\cos 2\mu - \cosh 2\lambda) , \\ (A + B)(\cosh^2 \lambda - \cos^2 \mu) &= a_1(\cos 2\mu + \cosh 2\lambda) + 2a_0 , \\ C \cosh^2 \lambda \cos^2 \mu &= a_0 + a_1(1 + \cosh 2\lambda + \cos 2\mu) \\ &\qquad + c_0 \cosh^2 \lambda \cos^2 \mu , \end{aligned}\right\} \qquad (616)$$

where we have written

$$a_0 = -2i\kappa_{17}; \quad 2a_1 = -i\kappa_{15}; \quad c_0 = i\kappa_{16} . \qquad (617)$$

From equations (616) we finally obtain

$$\left.\begin{aligned} a &= A(\cosh^2\lambda - \cos^2\mu) = a_0 + a_1 \cos 2\mu\,, \\ b &= B(\cosh^2\lambda - \cos^2\mu) = a_0 + a_1 \cosh 2\lambda\,, \\ c &= C \cosh^2\lambda \cos^2\mu \quad\;\; = a_0 + a_1(1 + \cosh 2\lambda + \cos 2\mu) \\ &\qquad\qquad\qquad\qquad\qquad + c_0 \cosh^2\lambda \cos^2\mu\,. \end{aligned}\right\} \tag{618}$$

Turning next to a consideration of the motions of the local centroids, we have, remembering that we have (arbitrarily) assumed that $f = g = h = 0$,

$$\left.\begin{aligned} a\Lambda_0 &= \Delta_1 = \xi\sqrt{\cosh^2\lambda - \cos^2\mu}\,, \\ bM_0 &= \Delta_2 = \eta\sqrt{\cosh^2\lambda - \cos^2\mu}\,, \\ cN_0 &= \Delta_3 = \zeta \cosh\lambda \cos\mu\,, \end{aligned}\right\} \tag{619}$$

where ξ, η, and ζ are given by equations (592). In (619), Λ_0, M_0, and N_0 are the components of the motion of the local centroid along the principal directions at (λ, μ, ν).

If we assume that the axis of the helical symmetry of $\mathfrak{B}$ coincides with the z-axis, then, according to equations (608) and (619),

$$\left.\begin{aligned} \Lambda_0 &= \frac{2k_3 \cosh\lambda \sin\mu}{(a_0 + a_1 \cos 2\mu)\sqrt{\cosh^2\lambda - \cos^2\mu}}\,, \\ M_0 &= \frac{2k_3 \sinh\lambda \cos\mu}{(a_0 + a_1 \cosh 2\lambda)\sqrt{\cosh^2\lambda - \cos^2\mu}}\,, \\ N_0 &= \frac{2k_6 \cosh\lambda \cos\mu}{a_0 + a_1(1 + \cosh 2\lambda + \cos 2\mu) + c_0 \cosh^2\lambda \cos^2\mu}\,. \end{aligned}\right\} \tag{620}$$

If we now introduce the further assumption that $\mathfrak{B}$ has an axial symmetry about the z-axis, then, as we have seen in § 23, $k_3 = 0$. In this case,

$$\left.\begin{aligned} \Lambda_0 &= M_0 = 0\,; \\ N_0 &= \frac{2k_6 \cosh\lambda \cos\mu}{a_0 + a_1(1 + \cosh 2\lambda + \cos 2\mu) + c_0 \cosh^2\lambda \cos^2\mu}\,. \end{aligned}\right\} \tag{621}$$

Equation (621) implies that the differential motions have now a purely rotational character.

It should be particularly noticed in this connection that the assumptions (1) $F = G = H = 0$ and (2) the axis of helical symmetry of $\mathfrak{B}$ coincides with the axis of symmetry of the "velocity surfaces"[20] still do not reduce the differential motions to a purely rotational kind. It is only when we introduce the further assumption that $\mathfrak{B}$ has an axial symmetry as well that the reduction to a pure rotation results.[21]

25. *Systems with spheroidal symmetry.*—We shall define a stellar system as having "spheroidal symmetry" if

$$\frac{\partial \mathfrak{B}}{\partial \mu} = \frac{\partial \mathfrak{B}}{\partial \nu} = 0 \; ; \qquad \frac{d\mathfrak{B}}{d\lambda} \neq 0 \, . \tag{622}$$

Equations (622) imply that *the equipotential surfaces form a system of confocal spheroids.* Since the potential function $\mathfrak{B}$ has now an axial symmetry, the solution for the Δ's is given by (610).

For the case (622) the compatibility conditions (IV_6) reduce to

$$\left.\begin{aligned} P^2 A \frac{d\mathfrak{B}}{d\lambda} &= -\frac{1}{2}\frac{\partial \chi}{\partial \lambda} \, , \\ P^2 H \frac{d\mathfrak{B}}{d\lambda} &= -\frac{1}{2}\frac{\partial \chi}{\partial \mu} \, , \\ R^2 G \frac{d\mathfrak{B}}{d\lambda} &= -\frac{1}{2}\frac{\partial \chi}{\partial \nu} \, . \end{aligned}\right\} \tag{623}$$

Equations (623) imply the following three integrability conditions:

$$\left.\begin{aligned} \frac{\partial}{\partial \mu}\left(P^2 A \frac{d\mathfrak{B}}{d\lambda}\right) &= \frac{\partial}{\partial \lambda}\left(P^2 H \frac{d\mathfrak{B}}{d\lambda}\right) , \\ \frac{\partial}{\partial \nu}\left(P^2 H \frac{d\mathfrak{B}}{d\lambda}\right) &= \frac{\partial}{\partial \mu}\left(R^2 G \frac{d\mathfrak{B}}{d\lambda}\right) , \\ \frac{\partial}{\partial \lambda}\left(R^2 G \frac{d\mathfrak{B}}{d\lambda}\right) &= \frac{\partial}{\partial \nu}\left(P^2 A \frac{d\mathfrak{B}}{d\lambda}\right) . \end{aligned}\right\} \tag{624}$$

[20] Which, under assumption (1), now exist.

[21] It is when we compare the manner in which known results (Eq. [621]) are derived here, as very special cases of the more general theory, with the methods and results of earlier investigators (cf. G. L. Clark, *M.N.*, **97**, 182, 1937) that the power and generality of our present theory becomes explicit.

Equations (624) are seen to be equivalent to

$$\left.\begin{aligned}\left\{\frac{\partial}{\partial\mu}(P^2A)-\frac{\partial}{\partial\lambda}(P^2H)\right\}\frac{d\mathfrak{B}}{d\lambda}&=P^2H\frac{d^2\mathfrak{B}}{d\lambda^2}\,,\\ \left\{\frac{\partial}{\partial\nu}(P^2A)-\frac{\partial}{\partial\lambda}(R^2G)\right\}\frac{d\mathfrak{B}}{d\lambda}&=R^2G\frac{d^2\mathfrak{B}}{d\lambda^2}\,,\\ \frac{\partial}{\partial\nu}(P^2H)&=\frac{\partial}{\partial\mu}(R^2G)\,.\end{aligned}\right\}\tag{625}$$

The conditions (625) will be identically satisfied only if

$$\frac{\partial}{\partial\mu}(P^2A)=0\,;\qquad\frac{\partial}{\partial\nu}(P^2A)=0\,;\qquad H=G=0\,.\tag{626}$$

From the general solution obtained in § 21 we readily verify that the conditions (626) imply that all the constants of integration in (552) except κ_{16} and κ_{17} are zero. The solution for the present case can therefore be obtained from that given in § 24 (Eq. [618]) by putting $a_1 = 0$. We thus have

$$\left.\begin{aligned}a&=b=\text{constant}=a_0\,,\\ c&=a_0+c_0\cosh^2\lambda\cos^2\mu\,.\end{aligned}\right\}\tag{627}$$

For the components $(\Lambda_0, \mathrm{M}_0, \mathrm{N}_0)$ of the motion of the local centroid along the principal directions at (λ, μ, ν) we have (cf. Eq. [621])

$$\Lambda_0=\mathrm{M}_0=0;\qquad \mathrm{N}_0=\frac{2k_6\cosh\lambda\cos\mu}{a_0+c_0\cosh^2\lambda\cos^2\mu}\,;\tag{628}$$

i.e., the motions are rotations about the axis of symmetry. We thus see that the restriction that the equipotential surfaces are confocal spheroids is very stringent indeed.

26. *The compatibility conditions.*—In §§ 21, 22, and 23 we saw that the variables which enable a symmetrical treatment of the fundamental equations are

$$p=i\lambda+\mu\,;\quad q=i\lambda-\mu\,.\tag{629}$$

It is therefore convenient to have the compatibility conditions (IV_6) written in terms of these variables.

The first two equations (IV_6) are

$$A \frac{\partial \mathfrak{B}}{\partial \lambda} + H \frac{\partial \mathfrak{B}}{\partial \mu} + G \frac{\partial \mathfrak{B}}{\partial \nu} = -\frac{1}{2P^2} \frac{\partial \chi}{\partial \lambda}, \tag{630}$$

$$H \frac{\partial \mathfrak{B}}{\partial \lambda} + B \frac{\partial \mathfrak{B}}{\partial \mu} + F \frac{\partial \mathfrak{B}}{\partial \nu} = -\frac{1}{2P^2} \frac{\partial \chi}{\partial \mu}. \tag{631}$$

Multiply equations (630) and (631) by $-i$ and 1, respectively, and add. We obtain

$$\left.\begin{aligned} \tfrac{1}{2}(A + B)\left(-i \frac{\partial \mathfrak{B}}{\partial \lambda} + \frac{\partial \mathfrak{B}}{\partial \mu}\right) + \{\tfrac{1}{2}(A - B) + iH\}\left(-i \frac{\partial \mathfrak{B}}{\partial \lambda} - \frac{\partial \mathfrak{B}}{\partial \mu}\right) \\ + (-iG + F) \frac{\partial \mathfrak{B}}{\partial \nu} = -\frac{1}{2P^2}\left(-i \frac{\partial \chi}{\partial \lambda} + \frac{\partial \chi}{\partial \mu}\right). \end{aligned}\right\} \tag{632}$$

Using equations (436), (437), (445′), and (471), the foregoing equation can be reduced to the form

$$(A + B) \frac{\partial \mathfrak{B}}{\partial p} + 4iH_2 \frac{\partial \mathfrak{B}}{\partial q} - i\psi \frac{\partial \mathfrak{B}}{\partial \nu} = -\frac{1}{P^2} \frac{\partial \chi}{\partial p}. \tag{633}$$

Similarly, by multiplying equations (630) and (631) by $-i$ and -1, respectively, and adding, we find, after some minor reductions, that

$$-4iH_1 \frac{\partial \mathfrak{B}}{\partial p} + (A + B) \frac{\partial \mathfrak{B}}{\partial q} + i\phi \frac{\partial \mathfrak{B}}{\partial \nu} = -\frac{1}{P^2} \frac{\partial \chi}{\partial q}. \tag{634}$$

Finally, the third of the compatibility conditions (IV_6) is readily seen to be equivalent to

$$-i\phi \frac{\partial \mathfrak{B}}{\partial p} + i\psi \frac{\partial \mathfrak{B}}{\partial q} + C \frac{\partial \mathfrak{B}}{\partial \nu} = -\frac{1}{2R^2} \frac{\partial \chi}{\partial \nu}. \tag{635}$$

Remembering that

$$\left.\begin{aligned} P^2 &= \cosh^2 \lambda - \cos^2 \mu = -\sin p \sin q, \\ R &= \cosh \lambda \cos \mu = \tfrac{1}{2}(\cos p + \cos q), \end{aligned}\right\} \tag{636}$$

we can combine equations (633), (634), and (635) in the single matrix equation

$$\begin{pmatrix} A+B & 4iH_2 & -i\psi \\ -4iH_1 & A+B & i\phi \\ -i\phi & i\psi & C \end{pmatrix} \begin{pmatrix} \dfrac{\partial \mathfrak{B}}{\partial p} \\ \dfrac{\partial \mathfrak{B}}{\partial q} \\ \dfrac{\partial \mathfrak{B}}{\partial \nu} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sin p \sin q} \dfrac{\partial \chi}{\partial p} \\ \dfrac{1}{\sin p \sin q} \dfrac{\partial \chi}{\partial q} \\ -\dfrac{2}{(\cos p + \cos q)^2} \dfrac{\partial \chi}{\partial \nu} \end{pmatrix} . \tag{637}$$

For systems with axial symmetry about the z-axis the compatibility conditions take the simpler forms

$$\left.\begin{aligned} (A+B)\frac{\partial \mathfrak{B}}{\partial p} + 4iH_2 \frac{\partial \mathfrak{B}}{\partial q} &= \frac{1}{\sin p \sin q} \frac{\partial \chi}{\partial p} , \\ -4iH_1 \frac{\partial \mathfrak{B}}{\partial p} + (A+B)\frac{\partial \mathfrak{B}}{\partial q} &= \frac{1}{\sin p \sin q} \frac{\partial \chi}{\partial q} , \\ i\phi \frac{\partial \mathfrak{B}}{\partial p} - i\psi \frac{\partial \mathfrak{B}}{\partial q} &= \frac{2}{(\cos p + \cos q)^2} \frac{\partial \chi}{\partial \nu} . \end{aligned}\right\} \tag{638}$$

Finally, for systems with spheroidal symmetry we have, according to (445),

$$\frac{\partial \mathfrak{B}}{\partial p} = \frac{\partial \mathfrak{B}}{\partial q} . \tag{639}$$

From equations (638) and (639) we can rederive the results of § 25.

27. *Prolate spheroidal co-ordinates.*—A system of prolate spheroidal co-ordinates is given by

$$\left.\begin{aligned} x &= \sinh \lambda \sin \mu \cos \nu , \\ y &= \sinh \lambda \sin \mu \sin \nu , \\ z &= \cosh \lambda \cos \mu . \end{aligned}\right\} \tag{640}$$

From (640) we have

$$\frac{x^2 + y^2}{\sinh^2 \lambda} + \frac{z^2}{\cosh^2 \lambda} = 1 \tag{641}$$

and

$$\frac{z^2}{\cos^2\mu} - \frac{x^2+y^2}{\sin^2\mu} = 1 . \tag{642}$$

In other words, the surfaces of constant λ and the surfaces of constant μ form a system of confocal prolate spheroids and hyperbolids, respectively.

For the system of co-ordinates (640) we have

$$P^2 = Q^2 = \cosh^2\lambda - \cos^2\mu\,; \qquad R = \sinh\lambda\sin\mu\,. \tag{643}$$

The fundamental differential equations in prolate spheroidal co-ordinates are very similar to those in oblate spheroidal co-ordinates. The solutions for A, B, C, F, G, H, ξ, η, and ζ can be obtained by methods analogous to those used in §§ 21 and 22. We shall consequently give only the final results. We have

$$\left.\begin{aligned}
A - B &= -2i(H_1 - H_2)\,,\\
A + B &= \frac{4i}{\sin p\sin q}(h_0\cos p\cos q - h_1\cos q + h_2\cos p + h_5)\,,\\
C &= \frac{8i}{(\cos p - \cos q)^2}\left\{\frac{\partial^2}{\partial\nu^2}(H_1\sin^2 p) + (2\cos q - \cos p)(h_0\cos p - h_1)\right.\\
&\quad\left. + (h_2\cos p + h_5) + H_1\sin^2 p\right\} + \frac{4i}{(\cos p - \cos q)}\left(\frac{dg_2}{d\nu} - \frac{dg_1}{d\nu}\cos p\right)\\
&\quad + 4i\gamma\,,\\
H &= H_1 + H_2\,,\\
\phi &= iF - G = \frac{4\sin p}{\cos p - \cos q}\frac{\partial H_1}{\partial\nu} + \frac{1}{\sin p}(g_2 - g_1\cos p)\,,\\
\psi &= iF + G = -\frac{4\sin q}{\cos p - \cos q}\frac{\partial H_2}{\partial\nu} + \frac{1}{\sin q}(f_2 - f_1\cos q)\,,
\end{aligned}\right\} \tag{644}$$

where

$$p = i\lambda + \mu; \qquad q = i\lambda - \mu \tag{645}$$

and

$$\left.\begin{aligned}
H_1 &= \frac{2}{\sin^2 p}(h_3 - h_1\cos p) - h_0\,,\\
H_2 &= \frac{2}{\sin^2 q}(h_4 - h_2\cos q) + h_0\,.
\end{aligned}\right\} \tag{646}$$

In the foregoing equations h_0, h_1, h_2, h_3, h_4, h_5, f_1, f_2, g_1, g_2, and γ are all functions of ν only and are given by

$$\left.\begin{aligned}
4h_0 &= -\tfrac{1}{2}(\kappa_{13}\sin 2\nu - \kappa_{14}\cos 2\nu) + \kappa_{15}\,,\\
h_1 &= \tfrac{1}{2}(\kappa_3\cos\nu + \kappa_4\sin\nu) + \tfrac{1}{2}(\kappa_7\cos 2\nu + \kappa_8\sin 2\nu) + \kappa_6\,,\\
h_2 &= \tfrac{1}{2}(\kappa_3\cos\nu + \kappa_4\sin\nu) - \tfrac{1}{2}(\kappa_7\cos 2\nu + \kappa_8\sin 2\nu) - \kappa_6\,,\\
h_3 &= \tfrac{1}{2}(\kappa_1\cos\nu + \kappa_2\sin\nu) + \tfrac{1}{2}(\kappa_{19}\cos 2\nu + \kappa_{20}\sin 2\nu) + \tfrac{1}{2}\kappa_{18}\,,\\
h_4 &= \tfrac{1}{2}(\kappa_1\cos\nu + \kappa_2\sin\nu) - \tfrac{1}{2}(\kappa_{19}\cos 2\nu + \kappa_{20}\sin 2\nu) - \tfrac{1}{2}\kappa_{18}\,,\\
h_5 &= (\kappa_{19}\cos 2\nu + \kappa_{20}\sin 2\nu) + \tfrac{1}{8}(\kappa_{13}\sin 2\nu - \kappa_{14}\cos 2\nu) + \kappa_{17}\,,\\
f_1 &= \tfrac{1}{2}(\kappa_{11}\cos\nu + \kappa_{12}\sin\nu) + \tfrac{1}{2}(\kappa_{13}\cos 2\nu + \kappa_{14}\sin 2\nu)\,,\\
g_1 &= \tfrac{1}{2}(\kappa_{11}\cos\nu + \kappa_{12}\sin\nu) - \tfrac{1}{2}(\kappa_{13}\cos 2\nu + \kappa_{14}\sin 2\nu)\,,\\
f_2 &= (\kappa_9\cos\nu + \kappa_{10}\sin\nu) - 2(-\kappa_7\sin 2\nu + \kappa_8\cos 2\nu) + \kappa_5\,,\\
g_2 &= (\kappa_9\cos\nu + \kappa_{10}\sin\nu) + 2(-\kappa_7\sin 2\nu + \kappa_8\cos 2\nu) - \kappa_5\,,\\
4\gamma &= \tfrac{1}{2}(\kappa_{13}\sin 2\nu - \kappa_{14}\cos 2\nu) - \kappa_{11}\sin\nu + \kappa_{12}\cos\nu + \kappa_{16}\,.
\end{aligned}\right\}\quad(647)$$

Further, the solutions for ξ, η, and ζ are found to be

$$\left.\begin{aligned}
\xi &= \frac{2k_3\sinh\lambda\cos\mu + 2(k_1\cos\nu + k_2\sin\nu)\cosh\lambda\sin\mu - (k_4\cos\nu + k_5\sin\nu)\sin 2\mu}{\cosh^2\lambda - \cos^2\mu}\,,\\
\eta &= \frac{-2k_3\cosh\lambda\sin\mu + 2(k_1\cos\nu + k_2\sin\nu)\sinh\lambda\cos\mu - (k_4\cos\nu + k_5\sin\nu)\sinh 2\lambda}{\cosh^2\lambda - \cos^2\mu}\,,\\
\zeta &= \frac{-2(k_1\sin\nu - k_2\cos\nu) + 2(-k_4\sin\nu + k_5\cos\nu)\cosh\lambda\cos\mu}{\sinh\lambda\sin\mu} + 2k_6\,.
\end{aligned}\right\}\quad(648)$$

If we choose the axis of helical symmetry of $\mathfrak{V}$ to coincide with the z-axis of our fundamental frame of reference, then it can be shown (cf. § 23) that $k_1 = k_2 = k_4 = k_5 = 0$, in which case we have the simpler expressions

$$\left.\begin{aligned}
\xi &= \frac{2k_3\sinh\lambda\cos\mu}{\cosh^2\lambda - \cos^2\mu}\,,\\
\eta &= -\frac{2k_3\cosh\lambda\sin\mu}{\cosh^2\lambda - \cos^2\mu}\,,\\
\zeta &= 2k_6\,.
\end{aligned}\right\}\quad(649)$$

The corresponding solutions for the Δ's are

$$\left.\begin{aligned}\Delta_1 &= \frac{2k_3 \sinh\lambda \cos\mu}{\sqrt{\cosh^2\lambda - \cos^2\mu}}\,,\\ \Delta_2 &= -\frac{2k_3 \cosh\lambda \sin\mu}{\sqrt{\cosh^2\lambda - \cos^2\mu}}\,,\\ \Delta_3 &= 2k_6 \sinh\lambda \sin\mu\,.\end{aligned}\right\} \tag{650}$$

Corresponding to the solutions (649) and (650), we have

$$\mathfrak{V}(x, y, z) = \mathfrak{V}\left(x^2 + y^2, z - \frac{k_3}{k_6}\tan^{-1}\frac{y}{x}\right). \tag{651}$$

If the helical symmetry about the z-axis degenerates into a simple axial symmetry, then $k_3 = 0$, and we have

$$\Delta_1 = \Delta_2 = 0; \qquad \Delta_3 = 2k_6 \sinh\lambda \sin\mu\,. \tag{652}$$

If, as in § 24, we consider the special case in which the principal axes of the velocity ellipsoid are assumed to be along the principal directions at the point considered, then we have (cf. Eq. [615])

$$\left.\begin{aligned}-(A - B)\sin p \sin q &= -i\kappa_{15} \sin p \sin q\,,\\ -(A - B)\sin p \sin q &= -i\kappa_{15} \cos p \cos q - 4i\kappa_{17}\,,\\ \tfrac{1}{4}C(\cos p - \cos q)^2 &= i\kappa_{15}\cos p \cos q - \tfrac{1}{2}i\kappa_{15} + 2i\kappa_{17}\\ &\quad + \tfrac{1}{4}i\kappa_{16}(\cos p - \cos q)^2\,.\end{aligned}\right\} \tag{653}$$

In terms of the variables λ and μ we find that equations (653) are equivalent to

$$\left.\begin{aligned}(A - B)(\cosh^2\lambda - \cos^2\mu) &= a_1(\cos 2\mu - \cosh 2\lambda)\,,\\ (A + B)(\cosh^2\lambda - \cos^2\mu) &= a_1(\cos 2\mu + \cosh 2\lambda) + 2a_0\,,\\ -C \sinh^2\lambda \sin^2\mu &= -a_0\\ &\quad + a_1(1 - \cosh 2\lambda - \cos 2\mu) - c_0 \sinh^2\lambda \sin^2\mu\,,\end{aligned}\right\} \tag{654}$$

where we have written

$$a_0 = -2i\kappa_{17}\,; \qquad 2a_1 = -i\kappa_{15}\,; \qquad c_0 = i\kappa_{16}\,. \tag{655}$$

From equations (654) we finally obtain

$$\left.\begin{aligned} a &= A(\cosh^2\lambda - \cos^2\mu) = a_0 + a_1\cos 2\mu\,,\\ b &= B(\cosh^2\lambda - \cos^2\mu) = a_0 + a_1\cosh 2\lambda\,,\\ c &= C\sinh^2\lambda\sin^2\mu = a_0 - a_1(1 - \cosh 2\lambda - \cos 2\mu)\\ &\qquad\qquad + c_0\sinh^2\lambda\sin^2\mu\,. \end{aligned}\right\} \tag{656}$$

If Λ_0, M_0, and N_0 are the components of the motion of the local centroid along the principal directions at (λ, μ, ν), then for the case under consideration

$$\left.\begin{aligned} a\Lambda_0 &= \Delta_1 = \xi\sqrt{\cosh^2\lambda - \cos^2\mu}\,,\\ bM_0 &= \Delta_2 = \eta\sqrt{\cosh^2\lambda - \cos^2\mu}\,,\\ cN_0 &= \Delta_3 = \zeta\sinh\lambda\sin\mu\,, \end{aligned}\right\} \tag{657}$$

where ξ, η, and ζ are given by (648). If we assume that the axis of helical symmetry of $\mathfrak{B}$ coincides with the z-axis, then, according to equations (650) and (656), we have

$$\left.\begin{aligned} \Lambda_0 &= \frac{2k_3\sinh\lambda\cos\mu}{(a_0 + a_1\cos 2\mu)\sqrt{\cosh^2\lambda - \cos^2\mu}}\,,\\ M_0 &= -\frac{2k_3\cosh\lambda\sin\mu}{(a_0 + a_1\cosh 2\lambda)\sqrt{\cosh^2\lambda - \cos^2\mu}}\,,\\ N_0 &= \frac{2k_6\sinh\lambda\sin\mu}{a_0 - a_1(1 - \cosh 2\lambda - \cos 2\mu) + c_0\sinh^2\lambda\sin^2\mu}\,. \end{aligned}\right\} \tag{658}$$

If we now introduce the further assumption that $\mathfrak{B}$ has an axial symmetry about the z-axis, then $k_3 = 0$, and we have

$$\left.\begin{aligned} &\Lambda_0 = M_0 = 0\,;\\ N_0 &= \frac{2k_6\sinh\lambda\sin\mu}{a_0 - a_1(1 - \cosh 2\lambda - \cos 2\mu) + c_0\sinh^2\lambda\sin^2\mu}\,. \end{aligned}\right\} \tag{659}$$

Equation (659) implies that the differential motions are now of a purely rotational kind.

Finally, if we consider systems with prolate spheroidal symmetry characterized by

$$\frac{\partial \mathfrak{B}}{\partial \nu} = \frac{\partial \mathfrak{B}}{\partial \mu} = 0 \; ; \qquad \frac{d\mathfrak{B}}{d\lambda} \neq 0 \, , \tag{660}$$

then, as in § 25, we find that $a_1 = 0$, and

$$\left.\begin{aligned} a &= b = \text{constant} = a_0 \, , \\ c &= a_0 + c_0 \sinh^2 \lambda \sin^2 \mu \, . \end{aligned}\right\} \tag{661}$$

Further, we have

$$\Lambda_0 = \mathrm{M}_0 = 0 \; ; \qquad \mathrm{N}_0 = \frac{2k_6 \sinh \lambda \sin \mu}{a_0 + c_0 \sinh^2 \lambda \sin^2 \mu} \, . \tag{662}$$

VII. CYLINDRICAL CO-ORDINATES

28. *The fundamental equations in cylindrical co-ordinates.*—If we choose a system of cylindrical co-ordinates, $(\bar{\omega}, \theta, z)$, in defining the fundamental frame of reference, we can let

$$\lambda = \bar{\omega}; \qquad \mu = \theta; \quad \nu = z \, . \tag{663}$$

Further, we have, as definitions,

$$x = \bar{\omega} \cos \theta; \qquad y = \bar{\omega} \sin \theta; \qquad z = z \, . \tag{664}$$

For this choice of co-ordinates

$$P = 1; \qquad Q = \omega; \qquad R = 1 \, . \tag{665}$$

The fundamental equations (I), (II), (III), and (IV) now take the forms

$$
\left.
\begin{aligned}
&\frac{\partial a}{\partial \varpi} = 0 \,, && \text{(i)}\\
&\frac{\partial b}{\partial \theta} + 2h = 0 \,, && \text{(ii)}\\
&\frac{\partial c}{\partial z} = 0 \,, && \text{(iii)}\\
&2\varpi \frac{\partial h}{\partial \varpi} + \frac{\partial a}{\partial \theta} - 2h = 0 \,, && \text{(iv)}\\
&2\frac{\partial g}{\partial \varpi} + \frac{\partial a}{\partial z} = 0 \,, && \text{(v)}\\
&2\frac{\partial h}{\partial \theta} + \varpi \frac{\partial b}{\partial \varpi} - 2(b-a) = 0 \,, && \text{(vi)}\\
&2\frac{\partial f}{\partial \theta} + \varpi \frac{\partial b}{\partial z} + 2g = 0 \,, && \text{(vii)}\\
&2\frac{\partial g}{\partial z} + \frac{\partial c}{\partial \varpi} = 0 \,, && \text{(viii)}\\
&2\varpi \frac{\partial f}{\partial z} + \frac{\partial c}{\partial \theta} = 0 \,, && \text{(ix)}\\
&f - \left(\varpi \frac{\partial f}{\partial \varpi} + \frac{\partial g}{\partial \theta} + \varpi \frac{\partial h}{\partial z}\right) = 0 \,, && \text{(x)}
\end{aligned}
\right\} \qquad (\mathrm{I}_7)
$$

$$
\left.
\begin{aligned}
&\frac{\partial \Delta_1}{\partial \varpi} = 0 \,, && \text{(i)}\\
&\frac{\partial \Delta_2}{\partial \theta} + \Delta_1 = 0 \,, && \text{(ii)}\\
&\frac{\partial \Delta_3}{\partial z} = 0 \,, && \text{(iii)}\\
&\varpi \frac{\partial \Delta_2}{\partial \varpi} + \frac{\partial \Delta_1}{\partial \theta} - \Delta_2 = 0 \,, && \text{(iv)}\\
&\frac{\partial \Delta_3}{\partial \theta} + \varpi \frac{\partial \Delta_2}{\partial z} = 0 \,, && \text{(v)}\\
&\frac{\partial \Delta_1}{\partial z} + \frac{\partial \Delta_3}{\partial \varpi} = 0 \,, && \text{(vi)}
\end{aligned}
\right\} \qquad (\mathrm{II}_7)
$$

$$\Delta_1 \frac{\partial \mathfrak{V}}{\partial \bar{\omega}} + \frac{\Delta_2}{\bar{\omega}} \frac{\partial \mathfrak{V}}{\partial \theta} + \Delta_3 \frac{\partial \mathfrak{V}}{\partial z} = 0 \,, \tag{III$_7$}$$

and

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \begin{pmatrix} \dfrac{\partial \mathfrak{V}}{\partial \bar{\omega}} \\ \dfrac{1}{\bar{\omega}} \dfrac{\partial \mathfrak{V}}{\partial \theta} \\ \dfrac{\partial \mathfrak{V}}{\partial z} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \dfrac{\partial \chi}{\partial \bar{\omega}} \\ \dfrac{1}{\bar{\omega}} \dfrac{\partial \chi}{\partial \theta} \\ \dfrac{\partial \chi}{\partial z} \end{pmatrix} . \tag{IV$_7$}$$

29. *The solution for the coefficients of the velocity ellipsoid.*—We shall first consider the system of equations (I_7). A simple examination of these equations reveals that

$$\left.\begin{aligned} &a \text{ is quadratic in } z \text{ and independent of } \bar{\omega} \,, \\ &b \text{ is quadratic in } z \,, \\ &c \text{ is independent of } z \,, \\ &f \text{ is linear in } z \,, \\ &g \text{ is linear in } z \text{ and linear in } \bar{\omega} \,, \\ &h \text{ is quadratic in } z \text{ and linear in } \bar{\omega} \,. \end{aligned}\right\} \tag{666}$$

According to (666), we can write

$$h = h_1(\theta, z)\bar{\omega} + h_2(\theta, z) \tag{667}$$

and

$$g = \{g_1(\theta)z + g_2(\theta)\}\bar{\omega} + g_3(\theta)z + g_4(\theta) \,. \tag{668}$$

Differentiating (vi) partially with respect to θ and using equations (i) and (ii), we find, after some simplifications,

$$\frac{\partial^2 h}{\partial \theta^2} + 4h - 3\bar{\omega} \frac{\partial h}{\partial \bar{\omega}} = 0 \,; \tag{669}$$

or, using (667), we have

$$\bar{\omega} \left(\frac{\partial^2 h_1}{\partial \theta^2} + h_1 \right) + \left(\frac{\partial^2 h_2}{\partial \theta^2} + 4h_2 \right) = 0 \,. \tag{670}$$

Since h_1 and h_2 are both independent of $\bar{\omega}$, we should have

$$\frac{\partial^2 h_1}{\partial \theta^2} + h_1 = 0 \; ; \qquad \frac{\partial^2 h_2}{\partial \theta^2} + 4h_2 = 0 \; , \tag{671}$$

or

$$\left.\begin{aligned} h_1 &= h_3(z) \cos \theta + h_4(z) \sin \theta \; , \\ h_2 &= h_5(z) \cos 2\theta + h_6(z) \sin 2\theta \; , \end{aligned}\right\} \tag{672}$$

where h_3, h_4, h_5, and h_6 are functions of z only. According to (666), these can be general functions quadratic in z. We can therefore write

$$h_n = h_{n0} + h_{n1} z + h_{n2} z^2 \qquad (n = 3, 4, 5, 6) \; , \tag{673}$$

where $h_{30}, \ldots , h_{62}$ are all constants, arbitrary in the first instance.

From equations (iv) and (667) we find

$$\frac{\partial a}{\partial \theta} = 2h_2 \; , \tag{674}$$

or, according to (672),

$$\frac{\partial a}{\partial \theta} = 2h_5 \cos 2\theta + 2h_6 \sin 2\theta \; . \tag{675}$$

Hence,

$$a = h_5 \sin 2\theta - h_6 \cos 2\theta + a_1(z) \; , \tag{676}$$

where, according to (666), we can write

$$a_1(z) = a_{10} + a_{11} z + a_{12} z^2 \; , \tag{677}$$

a_{10}, a_{11}, and a_{12} being constants.

From equations (ii), (667), and (672) we have

$$\frac{\partial b}{\partial \theta} = -\, 2(h_3 \cos \theta + h_4 \sin \theta)\, \bar{\omega} - 2(h_5 \cos 2\theta + h_6 \sin 2\theta) \; , \tag{678}$$

or, integrating,

$$b = -2(h_3 \sin\theta - h_4 \cos\theta)\bar{\omega} - (h_5 \sin 2\theta - h_6 \cos 2\theta) + b_1(\bar{\omega}, z). \quad (679)$$

Substituting for h, a, and b according to equations (667), (672), (676), and (679) in equation (vi), we find, after some minor reductions, that

$$\bar{\omega}\frac{\partial b_1}{\partial \bar{\omega}} - 2b_1 + 2a_1 = 0 . \quad (680)$$

Equation (680) can be integrated as it stands. We have

$$b_1 = b_2(z)\bar{\omega}^2 + a_1(z) , \quad (681)$$

where, according to (666), we can write

$$b_2(z) = b_{20} + b_{21}z + b_{22}z^2 . \quad (682)$$

According to equations (v) and (668), we have

$$2(g_1 z + g_2) = -\frac{\partial a}{\partial z} ; \quad (683)$$

or, substituting for a according to (676), we have

$$\left.\begin{aligned} 2g_1 z + 2g_2 = -(h_{51} + 2h_{52}z)\sin 2\theta + (h_{61} + 2h_{62}z)\cos 2\theta \\ - (a_{11} + 2a_{12}z) . \end{aligned}\right\} \quad (684)$$

Equating the terms which occur as coefficients of z and those which do not involve z, we have, respectively,

$$g_1 = -h_{52} \sin 2\theta + h_{62} \cos 2\theta - a_{12} , \quad (685)$$

$$g_2 = -\tfrac{1}{2}h_{51} \sin 2\theta + \tfrac{1}{2}h_{61} \cos 2\theta - \tfrac{1}{2}a_{11} . \quad (686)$$

Consider next equation (viii). According to (668), we have

$$\frac{\partial c}{\partial \bar{\omega}} = -2(g_1\bar{\omega} + g_3) ; \quad (687)$$

or, integrating and substituting for g_1 from (685), we have

$$c = (h_{52} \sin 2\theta - h_{62} \cos 2\theta + a_{12})\bar{\omega}^2 - 2g_3\bar{\omega} + c_1(\theta) , \quad (688)$$

where c_1 is a function of θ only. We have already seen that c is independent of z.

Differentiating equations (vii) and (ix) partially, with respect to z and θ, respectively, we have

$$2 \frac{\partial^2 f}{\partial z \partial \theta} + \bar{\omega} \frac{\partial^2 b}{\partial z^2} + 2 \frac{\partial g}{\partial z} = 0 \quad (689)$$

and

$$2\bar{\omega} \frac{\partial^2 f}{\partial \theta \partial z} + \frac{\partial^2 c}{\partial \theta^2} = 0 . \quad (690)$$

Combining equations (689) and (690), we have

$$\frac{\partial^2 c}{\partial \theta^2} = \bar{\omega}^2 \frac{\partial^2 b}{\partial z^2} + 2\bar{\omega} \frac{\partial g}{\partial z} ; \quad (691)$$

or, substituting for c, b, and g, we have

$$\left.\begin{aligned} &-4(h_{52} \sin 2\theta - h_{62} \cos 2\theta)\bar{\omega}^2 - 2\bar{\omega} \frac{d^2 g_3}{d\theta^2} + \frac{d^2 c_1}{d\theta^2} \\ &= \bar{\omega}^2\{-4(h_{32} \sin \theta - h_{42} \cos \theta)\bar{\omega} - 2(h_{52} \sin 2\theta - h_{62} \cos 2\theta) + 2b_{22}\bar{\omega}^2 + 2a_{12}\} \\ &+ 2\bar{\omega}^2(-h_{52} \sin 2\theta + h_{62} \cos 2\theta - a_{12}) + 2\bar{\omega} g_3 . \end{aligned}\right\} \quad (692)$$

From equation (692) it readily follows that

$$h_{32} = h_{42} = b_{22} = 0 \quad (693)$$

and

$$\frac{d^2 g_3}{d\theta^2} = - g_3 ; \qquad \frac{d^2 c_1}{d\theta^2} = 0 . \quad (694)$$

From (694) we have

$$g_3 = \gamma_1 \cos \theta + \gamma_2 \sin \theta , \quad (695)$$

$$c_1 = c_{10} + c_{11}\theta . \quad (696)$$

According to (666), f is linear in z. We can therefore write

$$f = f_1(\bar{\omega}, \theta)z + f_2(\bar{\omega}, \theta) . \tag{697}$$

From equation (ix) it follows that

$$2\bar{\omega}f_1 = -\frac{\partial c}{\partial \theta} ; \tag{698}$$

or, substituting for c according to equations (688), (695), and (696), we find

$$f_1 = -(h_{52} \cos 2\theta + h_{62} \sin 2\theta)\bar{\omega} - (\gamma_1 \sin \theta - \gamma_2 \cos \theta) - \frac{c_{11}}{2\bar{\omega}} . \tag{699}$$

Differentiating (x) partially with respect to z, we have, according to (697),

$$f_1 = \bar{\omega} \frac{\partial f_1}{\partial \bar{\omega}} + \frac{\partial}{\partial \theta} (g_1\bar{\omega} + g_3) + \bar{\omega} \frac{\partial^2 h}{\partial z^2} . \tag{700}$$

On substituting for f_1, g_1, g_3, and h in the foregoing equation, we find that

$$c_{11} = 0 . \tag{701}$$

Differentiating equation (x) partially with respect to θ and using (vii), we have

$$\bar{\omega} \frac{\partial b}{\partial z} + 2g + \bar{\omega} \frac{\partial}{\partial \bar{\omega}} \left(-\bar{\omega} \frac{\partial b}{\partial z} - 2g \right) + 2 \frac{\partial^2 g}{\partial \theta^2} + 2\bar{\omega} \frac{\partial^2 h}{\partial z \partial \theta} = 0 , \tag{702}$$

or, simplifying,

$$2g - \bar{\omega}^2 \frac{\partial^2 b}{\partial \bar{\omega} \partial z} - 2\bar{\omega} \frac{\partial g}{\partial \bar{\omega}} + 2 \frac{\partial^2 g}{\partial \theta^2} + 2\bar{\omega} \frac{\partial^2 h}{\partial z \partial \theta} = 0 . \tag{703}$$

On substituting for h, b, and g according to equations (667), (668), (672), (679), (681), (685), (686), and (693), we find that we are left with

$$-2b_{21}\bar{\omega}^3 + 2g_4 + 2 \frac{d^2 g_4}{d\theta^2} = 0 . \tag{704}$$

Hence,

$$b_{21} = 0 \; ; \qquad \frac{d^2 g_4}{d\theta^2} + g_4 = 0 \, . \tag{705}$$

Thus,

$$g_4 = \gamma_3 \cos\theta + \gamma_4 \sin\theta \, . \tag{706}$$

From equation (vii) we now obtain, after some reductions,

$$\left.\begin{aligned} \frac{\partial f_2}{\partial\theta} = \bar\omega^2(h_{31}\sin\theta - h_{41}\cos\theta) + \bar\omega(h_{51}\sin 2\theta - h_{61}\cos 2\theta) \\ - (\gamma_3\cos\theta + \gamma_4\sin\theta)\,, \end{aligned}\right\} \tag{707}$$

or, after integration,

$$\left.\begin{aligned} f_2 = \bar\omega^2(-h_{31}\cos\theta - h_{41}\sin\theta) + \tfrac{1}{2}(-h_{51}\cos 2\theta - h_{61}\sin 2\theta)\bar\omega \\ - (\gamma_3\sin\theta - \gamma_4\cos\theta) + f_3(\bar\omega)\,, \end{aligned}\right\} \tag{708}$$

where f_3 is a function of $\bar\omega$ only. Finally, from equation (x) we find that

$$f_3 = \bar\omega \frac{df_3}{d\bar\omega}\,, \tag{709}$$

or

$$f_3 = f_{30}\bar\omega \, . \tag{710}$$

We have now determined all the quantities in terms of $\bar\omega$, θ, and z. Collecting our results, we find the solutions in cylindrical co-ordinates for the coefficients of the velocity ellipsoid to be

$$\left.\begin{aligned} a &= h_5\sin 2\theta - h_6\cos 2\theta + a_1(z)\,, \\ b &= -2(h_3\sin\theta - h_4\cos\theta)\bar\omega - (h_5\sin 2\theta - h_6\cos 2\theta) + b_{20}\bar\omega^2 + a_1(z)\,, \\ c &= (h_{52}\sin 2\theta - h_{62}\cos 2\theta + a_{12})\bar\omega^2 - 2(\gamma_1\cos\theta + \gamma_2\sin\theta)\bar\omega + c_{10}\,, \\ f &= -\{(h_{52}\cos 2\theta + h_{62}\sin 2\theta)\bar\omega + (\gamma_1\sin\theta - \gamma_2\cos\theta)\}z \\ &\quad - (h_{31}\cos\theta + h_{41}\sin\theta)\bar\omega^2 - \tfrac{1}{2}(h_{51}\cos 2\theta + h_{61}\sin 2\theta)\bar\omega \\ &\quad - (\gamma_3\sin\theta - \gamma_4\cos\theta) + f_{30}\bar\omega\,, \\ g &= \{z(-h_{52}\sin 2\theta + h_{62}\cos 2\theta - a_{12}) - \tfrac{1}{2}(h_{51}\sin 2\theta - h_{61}\cos 2\theta + a_{11})\}\,\bar\omega \\ &\quad + (\gamma_1\cos\theta + \gamma_2\sin\theta)z + \gamma_3\cos\theta + \gamma_4\sin\theta\,, \\ h &= (h_3\cos\theta + h_4\sin\theta)\bar\omega + (h_5\cos 2\theta + h_6\sin 2\theta)\,, \end{aligned}\right\} \tag{711}$$

where

$$\left.\begin{aligned} h_3 &= h_{30} + h_{31}z\,, \\ h_4 &= h_{40} + h_{41}z\,, \\ h_5 &= h_{50} + h_{51}z + h_{52}z^2\,, \\ h_6 &= h_{60} + h_{61}z + h_{62}z^2\,, \\ a_1 &= a_{10} + a_{11}z + a_{12}z^2\,. \end{aligned}\right\} \tag{712}$$

The coefficients, $h_{30}, \ldots, a_{12}$, together with γ_1, γ_2, γ_3, γ_4, b_{20}, c_{10}, and f_{30}, are the twenty constants of integration.

30. *The solution for the motions of the local centroids.*—We shall now consider the six equations (II_7) for the Δ's. A simple examination of these equations reveals that

$$\left.\begin{aligned} &\Delta_1 \text{ is independent of } \bar{\omega} \text{ and linear in } z\,, \\ &\Delta_2 \text{ is linear in } \bar{\omega} \text{ and linear in } z\,, \\ &\Delta_3 \text{ is linear in } \bar{\omega} \text{ and independent of } z\,. \end{aligned}\right\} \tag{713}$$

Further, from equations (i) and (ii) we obtain

$$\frac{\partial^2 \Delta_2}{\partial \bar{\omega} \partial \theta} = 0\,. \tag{714}$$

Differentiating equation (iv) partially with respect to θ and using equations (ii) and (714), we are left with

$$\frac{\partial^2 \Delta_1}{\partial \theta^2} + \Delta_1 = 0\,. \tag{715}$$

Remembering that Δ_1 is linear in z and independent of $\bar{\omega}$, we can now write, according to (715), that

$$\Delta_1 = (\alpha_1 z + \beta_1)\cos\theta + (\alpha_2 z + \beta_2)\sin\theta\,, \tag{716}$$

where α_1, α_2, β_1, and β_2 are arbitrary constants. From equation (ii) we now obtain

$$\frac{\partial \Delta_2}{\partial \theta} = -(\alpha_1 z + \beta_1)\cos\theta - (\alpha_2 z + \beta_2)\sin\theta\,, \tag{717}$$

or, after integration,

$$\Delta_2 = -(\alpha_1 z + \beta_1) \sin \theta + (\alpha_2 z + \beta_2) \cos \theta + \delta_2(\bar{\omega}, z) . \tag{718}$$

Similarly, from equations (vi), (713), and (716) we obtain

$$\Delta_3 = -\bar{\omega}(\alpha_1 \cos \theta + \alpha_2 \sin \theta) + \delta_3(\theta) . \tag{719}$$

Substituting for Δ_2 and Δ_3 according to equations (718) and (719) in (v), we find

$$\frac{d\delta_3}{d\theta} + \bar{\omega} \frac{\partial \delta_2}{\partial z} = 0 . \tag{720}$$

Since δ_3 is independent of $\bar{\omega}$, it follows that

$$\frac{d\delta_3}{d\theta} = 0 ; \qquad \frac{\partial \delta_2}{\partial z} = 0 , \tag{721}$$

or

$$\delta_3 = \text{constant} = q \text{ (say)} ; \qquad \delta_2 \equiv \delta_2(\bar{\omega}) . \tag{722}$$

Finally, from equations (iv), (716), (718), and (722), we obtain

$$\bar{\omega} \frac{d\delta_2}{d\bar{\omega}} = \delta_2 , \tag{723}$$

or

$$\delta_2 = p\bar{\omega} , \tag{724}$$

where p is a constant. We have now solved the six equations (II_7). Collecting our results, we have

$$\left. \begin{aligned} \Delta_1 &= (\alpha_1 z + \beta_1) \cos \theta + (\alpha_2 z + \beta_2) \sin \theta , \\ \Delta_2 &= -(\alpha_1 z + \beta_1) \sin \theta + (\alpha_2 z + \beta_2) \cos \theta + p\bar{\omega} , \\ \Delta_3 &= -\bar{\omega}(\alpha_1 \cos \theta + \alpha_2 \sin \theta) + q . \end{aligned} \right\} \tag{725}$$

As in the previous cases, we see that the solution for the Δ's involve six arbitrary constants of integration.

31. *The discussion of the potential.*—According to (III_7), the linear partial differential equation for $\mathfrak{B}$ is

$$\Delta_1 \frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + \frac{\Delta_2}{\bar{\omega}} \frac{\partial \mathfrak{B}}{\partial \theta} + \Delta_3 \frac{\partial \mathfrak{B}}{\partial z} = 0 \,. \tag{726}$$

We shall first consider the case in which the z-axis is an axis of symmetry for $\mathfrak{B}$. Then

$$\frac{\partial \mathfrak{B}}{\partial \theta} = 0 \,, \tag{727}$$

and equation (726) reduces to

$$\Delta_1 \frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + \Delta_3 \frac{\partial \mathfrak{B}}{\partial z} = 0 \,. \tag{728}$$

From equation (728) it follows that

$$\frac{\Delta_1}{\Delta_3} = \frac{z(\alpha_1 \cos\theta + \alpha_2 \sin\theta) + \beta_1 \cos\theta + \beta_2 \sin\theta}{-\bar{\omega}(\alpha_1 \cos\theta + \alpha_2 \sin\theta) + q} \tag{729}$$

should be independent of θ. We readily verify that the expression on the right-hand side of (729) can be independent of θ in one of the two following ways only: either

$$q = 0 \,; \qquad \frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = -z_0 \text{ (say)} \qquad \text{(case [i])} \,, \tag{730}$$

or

$$q \neq 0 \,; \qquad \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \quad \text{(case [ii])} \,. \tag{731}$$

Consider first, case (i). According to equations (725) and (730), we now have

$$\left.\begin{aligned} \Delta_1 &= (z - z_0)(\alpha_1 \cos\theta + \alpha_2 \sin\theta) \,, \\ \Delta_2 &= (z - z_0)(-\alpha_1 \sin\theta + \alpha_2 \cos\theta) + p\bar{\omega} \,, \\ \Delta_3 &= -\bar{\omega}(\alpha_1 \cos\theta + \alpha_2 \sin\theta) \,. \end{aligned}\right\} \tag{732}$$

Equation (728) now reduces to

$$(z - z_0)\frac{\partial\mathfrak{B}}{\partial\bar{\omega}} - \bar{\omega}\frac{\partial\mathfrak{B}}{\partial z} = 0 . \tag{733}$$

The Lagrangian subsidiary equations for (733) are

$$\frac{d\bar{\omega}}{z - z_0} = -\frac{dz}{\bar{\omega}} . \tag{734}$$

Integrating (734), we have

$$\bar{\omega}^2 + (z - z_0)^2 = \text{constant} . \tag{735}$$

Hence, for the case under consideration

$$\mathfrak{B}(\bar{\omega}, \theta, z) \equiv \mathfrak{B}\{\bar{\omega}^2 + (z - z_0)^2\} . \tag{736}$$

Equation (736) implies that the system has a spherical symmetry about the point $(0, 0, z_0)$. Case (i) thus reduces to a problem we have already considered in Part V.

Considering next case (ii) (Eq. [731]), equations (725) now simplify to

$$\Delta_1 = 0 ; \qquad \Delta_2 = p\bar{\omega} ; \qquad \Delta_3 = q . \tag{737}$$

Equation (728) now reduces to

$$q\frac{\partial\mathfrak{B}}{\partial z} = 0 . \tag{738}$$

Hence, either

$$\frac{\partial\mathfrak{B}}{\partial z} = 0 \qquad \text{or} \qquad q = 0 . \tag{739}$$

The two subcases to which (731) leads are the following:

$$\frac{\partial\mathfrak{B}}{\partial\theta} = \frac{\partial\mathfrak{B}}{\partial z} = 0 ; \qquad \frac{d\mathfrak{B}}{d\bar{\omega}} \neq 0 , \tag{740}$$

in which case

$$\Delta_1 = 0 ; \qquad \Delta_2 = p\bar{\omega} ; \qquad \Delta_3 = q ; \tag{741}$$

and

$$\frac{\partial \mathfrak{B}}{\partial \theta} = 0 \; ; \qquad \frac{\partial \mathfrak{B}}{\partial z} \, , \; \frac{\partial \mathfrak{B}}{\partial \bar{\omega}} \neq 0 \, , \tag{742}$$

in which case

$$\Delta_1 = 0 \; ; \qquad \Delta_2 = p\bar{\omega} \; ; \qquad \Delta_3 = 0 \, . \tag{743}$$

Equation (742) corresponds to the general case of axial symmetry.

Returning to the general case, we have already shown in Part IV that stellar systems with differential motions must necessarily be characterized by an axis of helical symmetry. Let us choose the z-axis to be the axis of helical symmetry. For this choice of the z-axis, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$. For the partial differential equation of $\mathfrak{B}$ is then

$$p \frac{\partial \mathfrak{B}}{\partial \theta} + q \frac{\partial \mathfrak{B}}{\partial z} = 0 \, . \tag{744}$$

The corresponding Lagrangian subsidiary equations are

$$\frac{d\bar{\omega}}{0} = \frac{d\theta}{p} = \frac{dz}{q} \, . \tag{745}$$

Hence,

$$\mathfrak{B}(\bar{\omega}, \theta, z) \equiv \mathfrak{B}\left(\bar{\omega}, z - \frac{q}{p}\theta\right) \; ; \tag{746}$$

in other words, $\mathfrak{B}$ has a helical symmetry about the z-axis. If $q = 0$, we get stellar systems with simple axial symmetry about the z-axis; this agrees with equations (742) and (743). For the general case of helical symmetry about the z-axis we have

$$\Delta_1 = 0 \; ; \qquad \Delta_2 = p\bar{\omega} \; ; \qquad \Delta_3 = q \, . \tag{747}$$

The similarity of the foregoing solution (741) with those obtained in spheroidal co-ordinates (cf. Eqs. [608], [610], [650], and [652]) should be noticed.

32. *Stellar systems with $\partial \mathfrak{B}/\partial \theta = \partial \mathfrak{B}/\partial z = 0$.*—According to equations (740) and (741), for this case, we have

$$\Delta_1 = 0 \; ; \qquad \Delta_2 = p\bar{\omega} \; ; \qquad \Delta_3 = q \, . \tag{748}$$

Further, the compatibility conditions (IV_7) reduce to

$$\left.\begin{aligned} a\,\frac{d\mathfrak{B}}{d\bar{\omega}} &= -\frac{1}{2}\,\frac{\partial\chi}{\partial\bar{\omega}}\,, \\ h\bar{\omega}\,\frac{d\mathfrak{B}}{d\bar{\omega}} &= -\frac{1}{2}\,\frac{\partial\chi}{\partial\theta}\,, \\ g\,\frac{d\mathfrak{B}}{d\bar{\omega}} &= -\frac{1}{2}\,\frac{\partial\chi}{\partial z}\,. \end{aligned}\right\} \tag{749}$$

The equations (749) lead to the following integrability conditions:

$$\left.\begin{aligned} \frac{\partial}{\partial\theta}\left(a\,\frac{d\mathfrak{B}}{d\bar{\omega}}\right) &= \frac{\partial}{\partial\bar{\omega}}\left(h\bar{\omega}\,\frac{d\mathfrak{B}}{d\bar{\omega}}\right), \\ \frac{\partial}{\partial z}\left(h\bar{\omega}\,\frac{d\mathfrak{B}}{d\bar{\omega}}\right) &= \frac{\partial}{d\theta}\left(g\,\frac{d\mathfrak{B}}{d\bar{\omega}}\right), \\ \frac{\partial}{\partial\bar{\omega}}\left(g\,\frac{d\mathfrak{B}}{d\bar{\omega}}\right) &= \frac{\partial}{\partial z}\left(a\,\frac{d\mathfrak{B}}{d\bar{\omega}}\right). \end{aligned}\right\} \tag{750}$$

Equations (750) are readily seen to be equivalent to

$$\left.\begin{aligned} \left(\frac{\partial a}{\partial\theta} - \bar{\omega}\,\frac{\partial h}{\partial\bar{\omega}} - h\right)\frac{d\mathfrak{B}}{d\bar{\omega}} &= h\bar{\omega}\,\frac{d^2\mathfrak{B}}{d\bar{\omega}^2}\,, \\ \left(\frac{\partial a}{\partial z} - \frac{\partial g}{\partial\bar{\omega}}\right)\frac{d\mathfrak{B}}{d\bar{\omega}} &= g\,\frac{d^2\mathfrak{B}}{d\bar{\omega}^2}\,, \\ \bar{\omega}\,\frac{\partial h}{\partial z} &= \frac{\partial g}{\partial\theta}\,. \end{aligned}\right\} \tag{751}$$

We shall first consider the case when the equations (751) are identically satisfied. This will be the case if and only if

$$h = g = 0\,; \qquad \frac{\partial a}{\partial\theta} = \frac{\partial a}{\partial z} = 0\,. \tag{752}$$

According to (752) and the general solution for the coefficients of the velocity ellipsoid obtained in § 29 (Eqs. [711] and [712]), we now have

$$a = a_{10}\,; \qquad b = a_{10} + b_{20}\bar{\omega}^2\,; \qquad c = c_{10} \tag{753}$$

and

$$f = f_{30}\bar{\omega} \; ; \qquad h = g = 0 \, . \tag{754}$$

Equations (753) and (754) imply that one of the principal axes of the velocity ellipsoid is in the radial Π-direction; in other words, there is no deviation of the vertex in the radial direction. There is, however, a deviation of the vertex in the transverse plane. (See Fig. 2.) If ϵ denotes the angle which a principal axis in the trans-

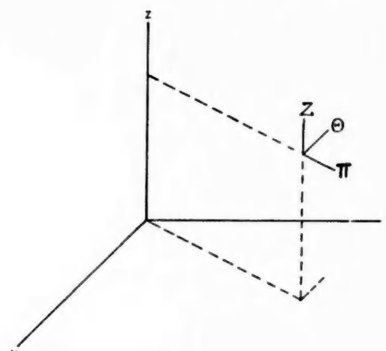

FIG. 2

verse plane makes with the positive transverse Θ-direction, then ϵ is given by

$$\tan 2\epsilon = \frac{2f}{b - c} \, , \tag{755}$$

or, according to equations (753) and (754),

$$\tan 2\epsilon = \frac{2f_{30}\bar{\omega}}{(a_{10} - c_{10}) + b_{20}\bar{\omega}^2} \, . \tag{756}$$

Let Π_0, Θ_0, and Z_0 denote the components of the motion of the local centroid along the principal directions at $(\bar{\omega}, \theta, z)$. Then, according to the definition of the Δ's and equations (748), (753), and (754), we have

$$a_{10}\Pi_0 = \Delta_1 = 0 \, , \tag{757}$$

$$(a_{10} + b_{20}\bar{\omega}^2)\Theta_0 + f_{30}\bar{\omega}Z_0 = \Delta_2 = p\bar{\omega} \, , \tag{758}$$

and

$$f_{30}\bar{\omega}\Theta_0 + c_{10}Z_0 = \Delta_3 = q\,. \tag{759}$$

Solving for Π_0, Θ_0, and Z_0, we have

$$\left.\begin{aligned} \Pi_0 &= 0\,, \\ \Theta_0 &= \frac{\bar{\omega}(c_{10}p - f_{30}q)}{c_{10}(a_{10} + b_{20}\bar{\omega}^2) - f_{30}^2\bar{\omega}^2}\,, \\ Z_0 &= \frac{q(a_{10} + b_{20}\bar{\omega}^2) - f_{30}p\bar{\omega}^2}{c_{10}(a_{10} + b_{20}\bar{\omega}^2) - f_{30}^2\bar{\omega}^2}\,. \end{aligned}\right\} \tag{760}$$

Equations (760) show that, although there is no component along the radial direction for the motion of the local centroid, we yet have motions in the transverse and in the Z-directions.

We shall now consider if there are other special forms for $\mathfrak{B}(\bar{\omega})$ for which the conditions (751) are satisfied.

Using equations (667) and (674), the first of the equations (751) can be transformed to

$$\frac{d^2\mathfrak{B}}{d\bar{\omega}^2}\bigg/\frac{d\mathfrak{B}}{d\bar{\omega}} = \frac{h_2 - 2h_1\bar{\omega}}{\bar{\omega}(h_1\bar{\omega} + h_2)}\,. \tag{761}$$

Using equation (v) of (I_7), the second of the equations (751) can be reduced to

$$\frac{d^2\mathfrak{B}}{d\bar{\omega}^2}\bigg/\frac{d\mathfrak{B}}{d\bar{\omega}} = -3\,\frac{\partial g}{\partial\bar{\omega}}\bigg/g\,. \tag{762}$$

Finally, when we use the general solutions for h and g obtained in § 29, the last of the equations (751) becomes

$$\left.\begin{aligned} &(h_{31}\cos\theta + h_{41}\sin\theta)\bar{\omega}^2 + \{(h_{51} + 2h_{52}z)\cos 2\theta + (h_{61} + 2h_{62}z)\sin 2\theta\}\bar{\omega} \\ &\quad = \{z(-2h_{52}\cos 2\theta - 2h_{62}\sin 2\theta) - (h_{51}\cos 2\theta + h_{61}\sin 2\theta)\}\bar{\omega} \\ &\quad\quad + (-\gamma_1\sin\theta + \gamma_2\cos\theta)z + (-\gamma_3\sin\theta + \gamma_4\cos\theta)\,. \end{aligned}\right\} \tag{763}$$

From equation (763) it readily follows that

$$h_{31} = h_{41} = h_{51} = h_{61} = h_{52} = h_{62} = \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0\,. \tag{764}$$

According to (764), the solution for the coefficients of the velocity ellipsoid now takes the simpler form

$$\left.\begin{aligned} a &= h_{50} \sin 2\theta - h_{60} \cos 2\theta + a_{10} + a_{11} z + a_{12} z^2 , \\ b &= -2(h_{30} \sin \theta - h_{40} \cos \theta)\bar{\omega} - (h_{50} \sin 2\theta - h_{60} \cos 2\theta) \\ &\qquad + b_{20}\bar{\omega}^2 + a_{10} + a_{11} z + a_{12} z^2 , \\ c &= a_{12}\bar{\omega}^2 + c_{10} , \\ f &= f_{30}\bar{\omega} , \\ g &= (-a_{12} z - \tfrac{1}{2} a_{11})\,\bar{\omega} , \\ h &= h_1\bar{\omega} + h_2 = (h_{30} \cos \theta + h_{40} \sin \theta)\bar{\omega} \\ &\qquad + (h_{50} \cos 2\theta + h_{60} \sin 2\theta) . \end{aligned}\right\} \tag{765}$$

Using (765), equation (762) takes the form

$$\frac{d^2\mathfrak{V}}{d\bar{\omega}^2} = -\frac{3}{\bar{\omega}} \frac{d\mathfrak{V}}{d\bar{\omega}} \qquad (a_{11}, a_{12} \neq 0) . \tag{766}$$

On the other hand, equation (761) requires that the right-hand side of (761) should be independent of θ. This can be arranged in one of two ways only: either $h_1 = 0$ or $h_2 = 0$. These two cases yield, respectively,

$$\frac{d^2\mathfrak{V}}{d\bar{\omega}^2} = \frac{1}{\bar{\omega}} \frac{d\mathfrak{V}}{d\bar{\omega}} ; \qquad h_1 = 0 \qquad (\text{case [i]}) \tag{767}$$

and

$$\frac{d^2\mathfrak{V}}{d\bar{\omega}^2} = -\frac{2}{\bar{\omega}} \frac{d\mathfrak{V}}{d\bar{\omega}} ; \qquad h_2 = 0 \qquad (\text{case [ii]}) . \tag{768}$$

From equations (766), (767), and (768) it follows that for either of the cases (i) and (ii), $g = 0$. Hence, the two possibilities which arise in this way are

$$a_{11} = a_{12} = h_{30} = h_{40} = 0 ; \qquad \frac{d\mathfrak{V}}{d\bar{\omega}} = \text{constant } \bar{\omega} \tag{769}$$

and

$$a_{11} = a_{12} = h_{50} = h_{60} = 0 \; ; \qquad \frac{d\mathfrak{V}}{d\bar{\omega}} = \frac{\text{constant}}{\bar{\omega}^2} \; . \tag{770}$$

Finally, a third case arises when $g \neq 0$ but $h = 0$. Then, according to (766),

$$\frac{d^2\mathfrak{V}}{d\bar{\omega}^2} = -\frac{3}{\bar{\omega}}\frac{d\mathfrak{V}}{d\bar{\omega}} \; ; \qquad h_1 = h_2 = 0 \; , \qquad g \neq 0 \qquad (\text{case [iii]}) \; ; \tag{771}$$

in other words, for case (iii)

$$h_{30} = h_{40} = h_{50} = h_{60} = 0 \; ; \qquad \frac{d\mathfrak{V}}{d\bar{\omega}} = \frac{\text{constant}}{\bar{\omega}^3} \; . \tag{772}$$

From (765) we can now write down the solutions for the coefficients of the velocity ellipsoid for the three cases (769), (770), and (772). They are:

$$\left.\begin{aligned} a &= h_{50} \sin 2\theta - h_{60} \cos 2\theta + a_{10} \; , \\ b &= -(h_{50} \sin 2\theta - h_{60} \cos 2\theta) + b_{20}\bar{\omega}^2 + a_{10} \; , \\ c &= c_{10} \; ; \qquad f = f_{30}\bar{\omega} \; ; \qquad g = 0 \; , \\ h &= h_{50} \cos 2\theta + h_{60} \sin 2\theta \; , \end{aligned}\right\} \text{case (i)} \tag{773}$$

$$\left.\begin{aligned} a &= a_{10} \; , \\ b &= -2(h_{30} \sin \theta - h_{40} \cos \theta)\bar{\omega} + b_{20}\bar{\omega}^2 + a_{10} \; , \\ c &= c_{10}; \qquad f = f_{30}\bar{\omega}; \qquad g = 0 \; , \\ h &= (h_{30} \cos \theta + h_{40} \sin \theta)\bar{\omega} \; , \end{aligned}\right\} \text{case (ii)} \tag{774}$$

and

$$\left.\begin{aligned} a &= a_{10} + a_{11}z + a_{12}z^2 \; , \\ b &= a_{10} + a_{11}z + a_{12}z^2 + b_{20}\bar{\omega}^2 \; , \\ c &= a_{12}\bar{\omega}^2 + c_{10} \; ; \qquad f = f_{30}\bar{\omega} \; ; \qquad h = 0 \; , \\ g &= -(a_{12}z + \tfrac{1}{2}a_{11})\bar{\omega} \; . \end{aligned}\right\} \text{case (iii)} \tag{775}$$

For these three special cases the motions of the local centroids are of a general kind. Further, we have illustrations here for the general phenomenon of the deviation of the vertex. This completes our

enumeration of the different possibilities for stellar systems which are characterized by

$$\frac{\partial \mathfrak{B}}{\partial \theta} = \frac{\partial \mathfrak{B}}{\partial z} = 0 \,. \tag{776}$$

33. *Stellar systems with* $\partial \mathfrak{B}/\partial \theta = 0$.—According to equations (742) and (743), for this case we have

$$\Delta_1 = 0 \,; \qquad \Delta_2 = p\bar{\omega} \,; \qquad \Delta_3 = 0 \,. \tag{777}$$

The compatibility conditions (IV_7) now reduce to

$$\left.\begin{aligned} a\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + g\,\frac{\partial \mathfrak{B}}{\partial z} &= -\frac{1}{2}\,\frac{\partial \chi}{\partial \bar{\omega}}\,, \\ h\bar{\omega}\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + f\bar{\omega}\,\frac{\partial \mathfrak{B}}{\partial z} &= -\frac{1}{2}\,\frac{\partial \chi}{\partial \theta}\,, \\ g\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + c\,\frac{\partial \mathfrak{B}}{\partial z} &= -\frac{1}{2}\,\frac{\partial \chi}{\partial z}\,. \end{aligned}\right\} \tag{778}$$

The equations (778) lead to the following integrability conditions:

$$\left.\begin{aligned} \frac{\partial}{\partial \theta}\left(a\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + g\,\frac{\partial \mathfrak{B}}{\partial z}\right) &= \frac{\partial}{\partial \bar{\omega}}\left(h\bar{\omega}\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + f\bar{\omega}\,\frac{\partial \mathfrak{B}}{\partial z}\right), \\ \frac{\partial}{\partial z}\left(h\bar{\omega}\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + f\bar{\omega}\,\frac{\partial \mathfrak{B}}{\partial z}\right) &= \frac{\partial}{\partial \theta}\left(g\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + c\,\frac{\partial \mathfrak{B}}{\partial z}\right), \\ \frac{\partial}{\partial \bar{\omega}}\left(g\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + c\,\frac{\partial \mathfrak{B}}{\partial z}\right) &= \frac{\partial}{\partial z}\left(a\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + g\,\frac{\partial \mathfrak{B}}{\partial z}\right). \end{aligned}\right\} \tag{779}$$

On carrying out the necessary differentiations and using (I_7), we can simplify the foregoing equations to take the forms

$$\left(h - 3\bar{\omega}\,\frac{\partial h}{\partial \bar{\omega}}\right)\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} - \bar{\omega}\left(2\,\frac{\partial f}{\partial \bar{\omega}} + \frac{\partial h}{\partial z}\right)\frac{\partial \mathfrak{B}}{\partial z} = h\bar{\omega}\,\frac{\partial^2 \mathfrak{B}}{\partial \bar{\omega}^2} + f\bar{\omega}\,\frac{\partial^2 \mathfrak{B}}{\partial \bar{\omega}\partial z}\,, \tag{780}$$

$$\left(\frac{\partial g}{\partial \theta} - \bar{\omega}\,\frac{\partial h}{\partial z}\right)\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} - 3\bar{\omega}\,\frac{\partial f}{\partial z}\,\frac{\partial \mathfrak{B}}{\partial z} = h\bar{\omega}\,\frac{\partial^2 \mathfrak{B}}{\partial \bar{\omega}\partial z} + f\bar{\omega}\,\frac{\partial^2 \mathfrak{B}}{\partial z^2}\,, \tag{781}$$

$$3\,\frac{\partial g}{\partial \bar{\omega}}\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} - 3\,\frac{\partial g}{\partial z}\,\frac{\partial \mathfrak{B}}{\partial z} = g\left(\frac{\partial^2 \mathfrak{B}}{\partial z^2} - \frac{\partial^2 \mathfrak{B}}{\partial \bar{\omega}^2}\right) + (a - c)\,\frac{\partial^2 \mathfrak{B}}{\partial \bar{\omega}\partial z}\,. \tag{782}$$

The general discussion of these equations is likely to be a difficult mathematical problem. We may notice, however, certain special solutions:

i) With no restriction on $\mathfrak{B}(\bar{\omega}, z)$, equations (780)-(782) are satisfied if

$$f = g = h = 0 ; \qquad a = c . \tag{783}$$

According to the general solution for the coefficients of the velocity ellipsoid obtained in § 29, we now have

$$a = a_{10} ; \qquad b = a_{10} + b_{20}\bar{\omega}^2 ; \qquad c = a_{10} . \tag{784}$$

From equation (777) we now obtain

$$\Pi_0 = 0 ; \qquad \Theta_0 = \frac{p\bar{\omega}}{a_{10} + b_{20}\bar{\omega}^2} ; \qquad Z_0 = 0 . \tag{785}$$

This is the Oort-Lindblad case.

ii) In case (i) the Π and the Z axes of the velocity ellipsoid are seen to be equal. This restriction can be avoided if

$$\frac{\partial^2 \mathfrak{B}}{\partial \bar{\omega} \partial z} = 0 . \tag{786}$$

The integrability conditions (780)-(782) can be satisfied if $f = g = h = 0$. We then have

$$a = a_{10} ; \qquad b = a_{10} + b_{20}\bar{\omega}^2 ; \qquad c = c_{10} . \tag{787}$$

Equation (785) continues to be valid for this case.[22]

34. *Spheroidal systems.*—As a special case of stellar systems with axial symmetry, we shall consider the case where the equipotential surfaces are a series of concentric spheroids.[23] This implies that

$$\mathfrak{B}(\bar{\omega}, z) \equiv \mathfrak{B}(\tau) , \tag{788}$$

[22] This case has been considered by Lindblad, among others (cf. *Bergstrands Festschrift*, p. 16).

[23] This case is, therefore, different from the one considered in § 25, where the equipotential surfaces were assumed to be a series of confocal spheroids.

where

$$\tau = \tfrac{1}{2}(\varpi^2 + \alpha z^2) = \tfrac{1}{2}(x^2 + y^2 + \alpha z^2) , \tag{789}$$

where α is an arbitrary real number.[24] When $\mathfrak{B}$ has the form (788), we have

$$\frac{\partial \mathfrak{B}}{\partial \varpi} = \varpi \frac{d\mathfrak{B}}{d\tau} ; \qquad \frac{\partial \mathfrak{B}}{\partial z} = \alpha z \frac{d\mathfrak{B}}{d\tau} \tag{790}$$

and

$$\left.\begin{aligned} \frac{\partial^2 \mathfrak{B}}{\partial \varpi^2} &= \frac{d\mathfrak{B}}{d\tau} + \varpi^2 \frac{d^2\mathfrak{B}}{d\tau^2} , \\ \frac{\partial^2 \mathfrak{B}}{\partial z^2} &= \alpha \frac{d\mathfrak{B}}{d\tau} + \alpha^2 z^2 \frac{d^2\mathfrak{B}}{d\tau^2} , \\ \frac{\partial^2 \mathfrak{B}}{\partial \varpi \partial z} &= \alpha \varpi z \frac{d^2\mathfrak{B}}{d\tau^2} . \end{aligned}\right\} \tag{791}$$

Substituting (790) and (791) in the integrability relations (780)–(782), we find

$$\left(3\varpi \frac{\partial h}{\partial \varpi} + 2\alpha z \frac{\partial f}{\partial \varpi} + \alpha z \frac{\partial h}{\partial z}\right)\frac{d\mathfrak{B}}{d\tau} = -\varpi(h\varpi + \alpha z f) \frac{d^2\mathfrak{B}}{d\tau^2} , \tag{792}$$

$$\left(\frac{\partial g}{\partial \theta} - \varpi \frac{\partial h}{\partial z} - 3\alpha z \frac{\partial f}{\partial z} - \alpha f\right)\frac{d\mathfrak{B}}{d\tau} = \alpha z(h\varpi + \alpha z f) \frac{d^2\mathfrak{B}}{d\tau^2} , \tag{793}$$

$$\left.\begin{aligned} &\left(3\varpi \frac{\partial g}{\partial \varpi} - 3\alpha z \frac{\partial g}{\partial z} - [\alpha - 1]g\right) \frac{d\mathfrak{B}}{d\tau} \\ &\qquad = (\alpha^2 z^2 g - \varpi^2 g + [a - c]\alpha \varpi z) \frac{d^2\mathfrak{B}}{d\tau^2} . \end{aligned}\right\} \tag{794}$$

Before we proceed to the general discussion of the foregoing equations, we shall first consider the case where

$$\frac{d^2\mathfrak{B}}{d\tau^2} = 0 , \tag{795}$$

or

$$\mathfrak{B} = \text{constant} + \text{constant } (\varpi^2 + \alpha z^2) . \tag{796}$$

[24] In practical cases $\alpha > 0$.

($\mathfrak{B}$ has therefore the same form as the expression for the potential inside a uniform spheroidal distribution of mass.) For this case the integrability conditions reduce to

$$3\bar{\omega}\frac{\partial h}{\partial\bar{\omega}} + 2\alpha z\frac{\partial f}{\partial\bar{\omega}} + \alpha z\frac{\partial h}{\partial z} = 0\,, \tag{797}$$

$$\frac{\partial g}{\partial\theta} - \bar{\omega}\frac{\partial h}{\partial z} - 3\alpha z\frac{\partial f}{\partial z} - \alpha f = 0\,, \tag{798}$$

$$3\bar{\omega}\frac{\partial g}{\partial\bar{\omega}} - 3\alpha z\frac{\partial g}{\partial z} - (\alpha - 1)g = 0\,. \tag{799}$$

We shall first discuss the foregoing equations for the case $\alpha \neq 1$. (The case $\alpha = 1$ will be considered separately.)

Consider first the equation (799). Since (cf. Eq. [668])

$$g = (g_1 z + g_2)\bar{\omega} + g_3 z + g_4\,, \tag{800}$$

where g_1, g_2, g_3, and g_4 are functions of θ only, we have

$$\left.\begin{aligned}3\bar{\omega}(g_1 z + g_2) - 3\alpha z(g_1\bar{\omega} + g_3)&\\ - (\alpha - 1)[(g_1 z + g_2)\bar{\omega} + g_3 z + g_4] &= 0\,.\end{aligned}\right\} \tag{801}$$

Equation (801) is readily seen to be equivalent to

$$4(1 - \alpha)g_1\bar{\omega}z + (4 - \alpha)g_2\bar{\omega} - (4\alpha - 1)g_3 z - (\alpha - 1)g_4 = 0\,. \tag{802}$$

Since $\alpha \neq 1$, we should have

$$g_1 = g_4 = 0\,; \qquad (4 - \alpha)g_2 = 0\,; \qquad (4\alpha - 1)g_3 = 0\,. \tag{803}$$

According to equations (685), (686), (695), and (706), we now have

$$h_{52} = h_{62} = a_{12} = \gamma_3 = \gamma_4 = 0 \tag{804}$$

and

$$\left.\begin{aligned}(4 - \alpha)(h_{51}\sin 2\theta - h_{61}\cos 2\theta + a_{11}) &= 0\,,\\ (4\alpha - 1)(\gamma_1\cos\theta + \gamma_2\sin\theta) &= 0\,.\end{aligned}\right\} \tag{805}$$

From equation (797) we obtain, on using the general solutions for the coefficients of the velocity ellipsoid obtained in § 29,

$$\left.\begin{aligned}3(h_{30}\cos\theta + h_{40}\sin\theta)\bar{\omega} + 2\alpha f_{30}z + 3(1-\alpha)h_{31}\bar{\omega}z\cos\theta&\\ + 3(1-\alpha)h_{41}\bar{\omega}z\sin\theta = 0\,.&\end{aligned}\right\}\quad(806)$$

Hence (since $\alpha \neq 1$),

$$h_{30} = h_{40} = f_{30} = h_{31} = h_{41} = 0\,. \quad (807)$$

Similarly, from equation (798) we obtain

$$\left.\begin{aligned}(\tfrac{1}{2}\alpha - 2)(h_{51}\cos 2\theta + h_{61}\sin 2\theta)\bar{\omega}&\\ + (4\alpha - 1)(\gamma_1\sin\theta - \gamma_2\cos\theta)\,z = 0\,,&\end{aligned}\right\}\quad(808)$$

or

$$\left.\begin{aligned}(4-\alpha)(h_{51}\cos 2\theta + h_{61}\sin 2\theta) &= 0\,,\\ (1-4\alpha)(\gamma_1\sin\theta - \gamma_2\cos\theta) &= 0\,.\end{aligned}\right\}\quad(809)$$

Combining equations (804), (805), (807), and (809), we have

$$h_{30} = h_{40} = h_{31} = h_{41} = h_{52} = h_{62} = a_{12} = f_{30} = \gamma_3 = \gamma_4 = 0\,, \quad (810)$$

$$(4-\alpha)h_{51} = (4-\alpha)h_{61} = (4-\alpha)a_{11} = 0\,, \quad (811)$$

$$(4\alpha - 1)\gamma_1 = (4\alpha - 1)\gamma_2 = 0\,. \quad (812)$$

From the foregoing equations it follows that we have to distinguish between the cases (i) $\alpha \neq 1$, 4, or $\frac{1}{4}$; (ii) $\alpha = 4$; and (iii) $\alpha = \frac{1}{4}$.

Case (i).—From equations (811) and (812) we have (since $\alpha \neq 4$ or $\frac{1}{4}$)

$$h_{51} = h_{61} = a_{11} = \gamma_1 = \gamma_2 = 0\,. \quad (813)$$

Hence, in the general solution (711) the only constants which do not vanish are h_{50}, h_{60}, a_{10}, b_{20}, and c_{10}. The solution for the coefficients of the velocity ellipsoid for this case is therefore given by

$$\left.\begin{aligned}a &= h_{50}\sin 2\theta - h_{60}\cos 2\theta + a_{10}\,,\\ b &= -h_{50}\sin 2\theta + h_{60}\cos 2\theta + a_{10} + b_{20}\bar{\omega}^2\,,\\ c &= c_{10}\,; \qquad f = g = 0\,,\\ h &= h_{50}\cos 2\theta + h_{60}\sin 2\theta\,.\end{aligned}\right\}\quad(814)$$

We thus see that there is a deviation of the vertex in the $\Pi\Theta$-plane. If ϵ denotes the angle which the vertex makes with the Π-direction, then, according to equation (814), we have

$$\tan 2\epsilon = \frac{2(h_{50}\cos 2\theta + h_{60}\sin 2\theta)}{2(h_{50}\sin 2\theta - h_{60}\cos 2\theta) - b_{20}\bar{\omega}^2}\,. \tag{815}$$

Considering next the motions (Π_0, Θ_0, Z_0) of the local centroids, we have (cf. Eq. [777])

$$\left.\begin{aligned} a\Pi_0 + h\Theta_0 &= 0\,,\\ h\Pi_0 + b\Theta_0 &= p\bar{\omega}\,,\\ cZ_0 &= 0\,. \end{aligned}\right\} \tag{816}$$

Hence, the motions of the local centroids have no components in the Z-direction.

Case (ii).—For this case $\alpha = 4$. Hence, according to equations (811) and (812),

$$\gamma_1 = \gamma_2 = 0; \qquad h_{51},\, h_{61},\, a_{11} \neq 0\,. \tag{817}$$

The solution for the coefficients of the velocity ellipsoid for this case can be written down from the general solution (711). We have

$$\left.\begin{aligned} a &= (h_{50} + h_{51}z)\sin 2\theta - (h_{60} + h_{61}z)\cos 2\theta + a_{10} + a_{11}z\,,\\ b &= -(h_{50} + h_{51}z)\sin 2\theta + (h_{60} + h_{61}z)\cos 2\theta + a_{10}\\ &\qquad\qquad + a_{11}z + b_{20}\bar{\omega}^2\,,\\ c &= c_{10}\,,\\ f &= -\tfrac{1}{2}(h_{51}\cos 2\theta + h_{61}\sin 2\theta)\bar{\omega}\,,\\ g &= -\tfrac{1}{2}(h_{51}\sin 2\theta - h_{61}\cos 2\theta + a_{11})\bar{\omega}\,,\\ h &= (h_{50} + h_{51}z)\cos 2\theta + (h_{60} + h_{61}z)\sin 2\theta\,. \end{aligned}\right\} \tag{818}$$

The motion (Π_0, Θ_0, Z_0) of the local centroid is given by

$$\left.\begin{aligned} a\Pi_0 + h\Theta_0 + gZ_0 &= 0\,,\\ h\Pi_0 + b\Theta_0 + fZ_0 &= p\bar{\omega}\,,\\ g\Pi_0 + f\Theta_0 + cZ_0 &= 0\,. \end{aligned}\right\} \tag{819}$$

The differential motions in this case are therefore of a quite general kind. In particular, the motions of the local centroids have components in the Z-direction as well.

Case (iii).—For this case, $\alpha = \frac{1}{4}$. According to equations (811) and (812), we have

$$h_{51} = h_{61} = a_{11} = 0; \qquad \gamma_1, \gamma_2 \neq 0 . \tag{820}$$

From the general solution (711) we now obtain

$$\left.\begin{aligned} a &= h_{50} \sin 2\theta - h_{60} \cos 2\theta + a_{10} , \\ b &= -h_{50} \sin 2\theta + h_{60} \cos 2\theta + a_{10} + b_{20}\bar{\omega}^2 , \\ c &= -2(\gamma_1 \cos\theta + \gamma_2 \sin\theta)\bar{\omega} + c_{10} , \\ f &= -(\gamma_1 \sin\theta - \gamma_2 \cos\theta)z , \\ g &= (\gamma_1 \cos\theta + \gamma_2 \sin\theta)z , \\ h &= h_{50} \cos 2\theta + h_{60} \sin 2\theta . \end{aligned}\right\} \tag{821}$$

The components of the motion of the local centroid Π_0, Θ_0, and Z_0 are given by equation (819) with a, b, c, f, g, and h defined as in equation (821).

The analysis has thus disclosed the existence of two critical cases ($\alpha = 4$ and $\alpha = \frac{1}{4}$) when the potential $\mathfrak{V}$ has the form (796). More explicitly, the expressions for $\mathfrak{V}$ for these two cases are

$$\mathfrak{V} = \text{constant} + \text{constant}(\bar{\omega}^2 + 4z^2) \tag{822}$$

and

$$\mathfrak{V} = \text{constant} + \text{constant}(\bar{\omega}^2 + \tfrac{1}{4}z^2) . \tag{823}$$

As we have already pointed out, the expression for the potential inside a homogeneous spheroid has the form (796). The constant α in (796) should therefore depend only on the ratio of the axes, κ, of the uniform spheroidal distribution of mass. From the known expression for $\mathfrak{V}$ inside a homogeneous spheroid[25] we readily find that the relations which determine κ in terms of α are

$$\alpha = \frac{2\vartheta}{1 - \vartheta} , \tag{824}$$

[25] See, e.g., E. J. Routh, *A Treatise on Analytical Statics*, **2**, 106–116, Cambridge, England, 1922.

where ϑ is a numerical constant related to κ by the relations

$$\vartheta = \frac{\kappa^2}{\kappa^2 - 1} - \frac{\kappa^2}{(\kappa^2 - 1)^{3/2}} \tan^{-1}\sqrt{\kappa^2 - 1} \qquad (\kappa > 1) \quad (825)$$

and

$$\vartheta = -\frac{\kappa^2}{1 - \kappa^2} + \frac{\kappa^2}{(1 - \kappa^2)^{3/2}} \log\left(\frac{1}{\kappa} + \sqrt{\frac{1}{\kappa} - 1}\right) \quad (\kappa < 1)\,. \quad (826)$$

From the relations (824) and (825) we find that, when $\alpha = 4$,

$$\vartheta = \tfrac{2}{3}; \qquad \kappa = 3.410\ldots\ldots \qquad (827)$$

The critical case $\alpha = 4$ arises, therefore, for stellar motions inside an oblate homogeneous spheroid which has its major axis (about) 3.410 times the minor axis. (A cross-section of this critical spheroid is shown in Figure 3.) It is of interest to recall in this connection

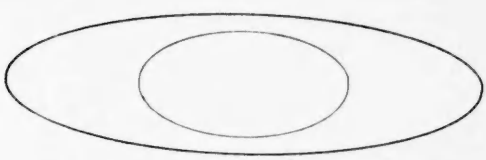

FIG. 3.—The critical oblate spheroid

that observations seem to indicate an upper limit to the eccentricities of elliptical nebulae. Indeed, according to Hubble,[26] "they [elliptical nebulae] range from globular objects through ellipsoidal figures to a limiting lenticular form with a ratio of the axes about 3 to 1." It is probable that the critical value for the ratio $\kappa = 3.410\ldots$, which, we have found, is in some way connected with the upper limit for the ratio of the axes for elliptical nebulae. The plausibility of this suggestion rests chiefly on the circumstance that for $1 < \kappa < 3.410\ldots$ there are no motions of the local centroids

[26] *The Realm of the Nebulae*, p. 39, New Haven, Yale University Press, 1936.

perpendicular to the galactic plane, while such motions appear suddenly for $\kappa = 3.410\ldots$.

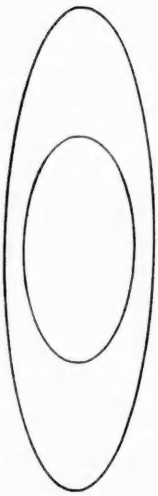

FIG. 4.—The critical prolate spheroid.

Considering next the case $\alpha = \frac{1}{4}$, we find from equations (824) and (826) that for this value of α

$$\vartheta = \tfrac{1}{9}; \qquad \kappa = 0.3142\ldots \tag{828}$$

In other words, the critical case $\alpha = \frac{1}{4}$ arises for stellar motions inside a prolate homogeneous spheroid which has its major axis (about) $3.183\ldots$ times its minor axis. A cross-section of this critical spheroid is shown in Figure 4.

The motions inside the two critical homogeneous spheroids have a certain general similarity. In particular, in both the cases the motions of the local centroids have components in the Z-direction. However, the two cases are really fundamentally different, as is apparent when we compare the solutions for the coefficients of the velocity ellipsoid for the two cases (Eqs. [818] and [821]).[27]

We shall now return to a general discussion of the equations (792)–(794). From (792) and (793) it follows that

$$\left.\begin{aligned} &az\left(3\bar{\omega}\frac{\partial h}{\partial\bar{\omega}} + 2az\frac{\partial f}{\partial\bar{\omega}} + az\frac{\partial h}{\partial z}\right) \\ &\qquad + \bar{\omega}\left(\frac{\partial g}{\partial\theta} - \bar{\omega}\frac{\partial h}{\partial z} - 3az\frac{\partial f}{\partial z} - af\right) = 0\,. \end{aligned}\right\} \tag{829}$$

Substituting for f, g, and h in (829), according to the general solution obtained in § 29, we find

$$\left.\begin{aligned} &(\alpha - 1)(h_{31}\cos\theta + h_{41}\sin\theta)\bar{\omega}^3 + 4(\alpha - 1)(h_{52}\cos 2\theta + h_{62}\sin 2\theta)\bar{\omega}^2 z \\ &\quad - 3\alpha(\alpha - 1)(h_{31}\cos\theta + h_{41}\sin\theta)\bar{\omega}z^2 + \tfrac{1}{2}(\alpha - 4)(h_{51}\cos 2\theta + h_{61}\sin 2\theta)\bar{\omega}^2 \\ &\quad - \alpha f_{30}\bar{\omega}^2 + 2\alpha^2 f_{30}z^2 + [\{3\alpha h_{30} + (1 - 4\alpha)\gamma_2\}\cos\theta \\ &\quad + \{3\alpha h_{40} + (4\alpha - 1)\gamma_1\}\sin\theta]\bar{\omega}z + (\alpha - 1)(\gamma_3\sin\theta - \gamma_4\cos\theta)\bar{\omega} = 0\,. \end{aligned}\right\} \tag{830}$$

[27] It is of interest to notice that, while the solution (818) for the case $\alpha = 4$ involves eight constants of integration, the solution (821) for the case $\alpha = \frac{1}{4}$ involves only seven constants of integration.

Remembering that we have excluded the case $\alpha = 1$, equation (830) implies that

$$h_{31} = h_{41} = h_{52} = h_{62} = f_{30} = \gamma_3 = \gamma_4 = 0 \, , \tag{831}$$

$$(\alpha - 4)h_{51} = (\alpha - 4)h_{61} = 0 \, , \tag{832}$$

and

$$3\alpha h_{30} = (4\alpha - 1)\gamma_2 \, ; \qquad 3\alpha h_{40} = (1 - 4\alpha)\gamma_1 = 0 \, . \tag{833}$$

Equation (792) is now found to take the form

$$\left.\begin{aligned}&\{(h_{30}\cos\theta + h_{40}\sin\theta)\bar{\omega}^2 + \alpha(\gamma_2\cos\theta - \gamma_1\sin\theta)z^2\\&\quad + (1 - \tfrac{1}{2}\alpha)(h_{51}\cos 2\theta + h_{61}\sin 2\theta)\bar{\omega}z\\&\quad + (h_{50}\cos 2\theta + h_{60}\sin 2\theta)\bar{\omega}\}\,\frac{d^2\mathfrak{B}}{d\tau^2}\\&\quad = -3(h_{30}\cos\theta + h_{40}\sin\theta)\,\frac{d\mathfrak{B}}{d\tau}\,.\end{aligned}\right\} \tag{834}$$

Since $\mathfrak{B}$ is a function of τ only, it follows that the ratio of the coefficients of $d\mathfrak{B}/d\tau$ and $d^2\mathfrak{B}/d\tau^2$ should also be a function of τ only. It is readily seen that this can be arranged only in either of the following two ways:

$$h_{30} = h_{40} = 0 \tag{835}$$

or

$$\left.\begin{aligned}h_{30} = \gamma_2; \qquad h_{40} = -\gamma_1;\\(1 - \tfrac{1}{2}\alpha)h_{51} = (1 - \tfrac{1}{2}\alpha)h_{61} = h_{50} = h_{60} = 0\,.\end{aligned}\right\} \tag{836}$$

Equation (835) leads to the case we have already considered, since, according to equation (834), we now have

$$\frac{d^2\mathfrak{B}}{d\tau^2} = 0\,. \tag{836$'$}$$

On the other hand, combining the equations (832), (833), and (836), we find that

$$h = f = 0\,. \tag{837}$$

In other words, if $d^2\mathfrak{V}/d\tau^2 \neq 0$, both the equations (792) and (793) must be satisfied identically. It now readily follows that $g = 0$ and $a = c = a_{10}$. We have thus shown that if $d^2\mathfrak{V}/d\tau^2 \neq 0$, then the integrability relations (792), (793), and (794) must be satisfied identically, and that

$$a = c = a_{10} \; ; \qquad b = a_{10} + b_{20}\bar{\omega}^2 \; ; \qquad f = g = h = 0 \; . \tag{838}$$

This case is therefore identical with the case (i) considered in § 33. Consequently, the restriction on $\mathfrak{V}$ that $\mathfrak{V}(\bar{\omega}, z) \equiv \mathfrak{V}(\tau)$ does not provide any essentially new features unless $d^2\mathfrak{V}/d\tau^2 = 0$. On the other hand, the assumption that the equipotential surfaces are concentric spheroids is of physical interest only for the case of the potential inside a homogeneous spheroidal distribution of mass, i.e., precisely when $d^2\mathfrak{V}/d\tau^2 = 0$.

35. *Spherical systems.*—The methods developed in § 34 enable us to give alternative derivations for some of the results obtained in Part V. For, when $\mathfrak{V}$ has spherical symmetry, we can write

$$\mathfrak{V}(\bar{\omega}, z) \equiv \mathfrak{V}\{\tfrac{1}{2}(\bar{\omega}^2 + z^2)\} = \mathfrak{V}(\tau) \; ; \tag{839}$$

i.e., $\alpha = 1$—the case which was expressly excluded in § 34. We shall now briefly discuss this case.

As in § 34, we shall first consider the case

$$\frac{d^2\mathfrak{V}}{d\tau^2} = 0 \; . \tag{840}$$

Equations (792)–(793) now require the validity of the relations (797)–(799) for the case $\alpha = 1$. From equation (799) we have

$$\bar{\omega}\frac{\partial g}{\partial \bar{\omega}} - z\frac{\partial g}{\partial z} = 0 \; . \tag{841}$$

Using equation (800) for g, we find

$$g_2\bar{\omega} - g_3 z = 0 \; , \tag{842}$$

or

$$g_2 = g_3 = 0 \; . \tag{843}$$

From the equations (686) and (695) for g_2 and g_3 we find

$$h_{51} = h_{61} = a_{11} = \gamma_1 = \gamma_2 = 0 . \tag{844}$$

Substituting in equation (797) for f and h, according to (711), and using (844), we obtain

$$3(h_{30} \cos \theta + h_{40} \sin \theta)\bar{\omega} + 2f_{30}z = 0 , \tag{845}$$

or

$$h_{30} = h_{40} = f_{30} = 0 . \tag{846}$$

It is easily verified that equation (798) is now satisfied. The solution for the coefficients of the velocity ellipsoid can now be written down. We have (cf. Eqs. [711], [844], and [846])

$$\left.\begin{aligned}
a &= (h_{50} + h_{52}z^2) \sin 2\theta - (h_{60} + h_{62}z^2) \cos 2\theta + a_{10} + a_{12}z^2 , \\
b &= -2(h_{31} \sin \theta - h_{41} \cos \theta)\bar{\omega}z - (h_{50} + h_{52}z^2) \sin 2\theta \\
&\qquad + (h_{60} + h_{62}z^2) \cos 2\theta + a_{10} + a_{12}z^2 + b_{20}\bar{\omega}^2 , \\
c &= (h_{52} \sin 2\theta - h_{62} \cos 2\theta + a_{12})\bar{\omega}^2 + c_{10} , \\
f &= -(h_{52} \cos 2\theta + h_{62} \sin 2\theta)\, \bar{\omega}z - (h_{31} \cos \theta + h_{41} \sin \theta)\bar{\omega}^2 \\
&\qquad - \gamma_3 \sin \theta + \gamma_4 \cos \theta , \\
g &= (-h_{52} \sin 2\theta + h_{62} \cos 2\theta - a_{12})\bar{\omega}z + \gamma_3 \cos \theta + \gamma_4 \sin \theta , \\
h &= (h_{31} \cos \theta + h_{41} \sin \theta)\, \bar{\omega}z + (h_{50} + h_{52}z^2) \cos 2\theta \\
&\qquad + (h_{60} + h_{62}z^2) \sin 2\theta .
\end{aligned}\right\} \tag{847}$$

Thus, when the potential has the form

$$\mathfrak{V} = \text{constant} + \text{constant } r^2 , \tag{848}$$

the solution for a, b, c, f, g, and h involves twelve arbitrary constants, in agreement with the results in § 19.

If we now wish to satisfy the integrability relations (792)–(794) identically (for the case $\alpha = 1$), then the coefficients of $d\mathfrak{V}/d\tau$ and $d^2\mathfrak{V}/d\tau^2$ in these equations must all vanish; i.e., in addition to (844) and (846) we should require that

$$h\bar{\omega} + zf = 0 \tag{849}$$

and

$$(z^2 - \bar{\omega}^2)g + (a - c)\bar{\omega}z = 0 . \tag{850}$$

We readily verify that equations (847), (849), and (850) imply that

$$h_{50} = h_{60} = \gamma_3 = \gamma_4 = 0 \tag{851}$$

and that

$$a_{10} = c_{10} . \tag{852}$$

Hence, when the equations (792)–(794) are satisfied identically for the case $\alpha = 1$, the solutions for a, b, c, f, g, and h are given by

$$\left.\begin{aligned}
a &= (h_{52} \sin 2\theta - h_{62} \cos 2\theta)z^2 + a_{10} + a_{12}z^2 , \\
b &= -\ 2(h_{31} \sin \theta - h_{41} \cos \theta)\bar{\omega}z - (h_{52} \sin 2\theta - h_{62} \cos 2\theta)z^2 \\
&\qquad\qquad + a_{10} + a_{12}z^2 + b_{20}\bar{\omega}^2 , \\
c &= (h_{52} \sin 2\theta - h_{62} \cos 2\theta + a_{12})\bar{\omega}^2 + a_{10} , \\
f &= -(h_{52} \cos 2\theta + h_{62} \sin 2\theta)\bar{\omega}z - (h_{31} \cos \theta + h_{41} \sin \theta)\bar{\omega}^2 , \\
g &= (-h_{52} \sin 2\theta + h_{62} \cos 2\theta - a_{12})\bar{\omega}z , \\
h &= (h_{31} \cos \theta + h_{41} \sin \theta)\bar{\omega}z + (h_{52} \cos 2\theta + h_{62} \sin 2\theta)z^2 .
\end{aligned}\right\} \tag{853}$$

The solution (853) involves seven arbitrary constants.

The general discussion of the relations (792)–(794) for the case $\alpha = 1$ proceeds on lines similar to that in § 34. Thus, from equation (830) we now conclude that

$$h_{51} = h_{61} = f_{30} = 0 \tag{854}$$

and

$$h_{30} = \gamma_2 ; \qquad h_{40} = -\ \gamma_1 . \tag{855}$$

Using equations (711), (854), and (855), equation (792) can be reduced to

$$\left.\begin{aligned}
&\{(h_{30} \cos \theta + h_{40} \sin \theta)(\bar{\omega}^2 + z^2) + (h_{50} \cos 2\theta + h_{60} \sin 2\theta)\bar{\omega} \\
&\quad - (\gamma_3 \sin \theta - \gamma_4 \cos \theta)z\} \frac{d^2\mathfrak{V}}{d\tau^2} = -3(h_{30} \cos \theta + h_{40} \sin \theta) \frac{d\mathfrak{V}}{d\tau} .
\end{aligned}\right\} \tag{856}$$

The ratio of the coefficients of $d^2\mathfrak{V}/d\tau^2$ and $d\mathfrak{V}/d\tau$ in the foregoing equation should be a function of τ only. This is possible in the following two cases only. Either

$$h_{30} = h_{40} = 0 , \tag{857}$$

in which case

$$\frac{d^2\mathfrak{V}}{d\tau^2} = 0 , \tag{858}$$

or

$$h_{50} = h_{60} = \gamma_3 = \gamma_4 = 0 , \tag{859}$$

in which case

$$\frac{d^2\mathfrak{V}}{d\tau^2} = -\frac{3}{\bar{\omega}^2 + z^2}\frac{d\mathfrak{V}}{d\tau} = -\frac{3}{2\tau}\frac{d\mathfrak{V}}{d\tau} . \tag{860}$$

We have already considered the case (858). On the other hand equation (860) yields

$$\frac{d\mathfrak{V}}{d\tau} = \frac{\text{constant}}{\tau^{3/2}} , \tag{861}$$

or

$$\frac{d\mathfrak{V}}{dr} = \frac{\text{constant}}{r^2} . \tag{862}$$

Using equations (711), (854), (855), and (859), we find that equation (794) can be reduced to

$$\left.\begin{aligned} \{(h_{30}\sin\theta - h_{40}\cos\theta)z(\bar{\omega}^2 + z^2) + (a_{10} - c_{10})\bar{\omega}z + \tfrac{1}{2}a_{11}\bar{\omega}(\bar{\omega}^2 + z^2)\}\,\frac{d^2\mathfrak{V}}{d\tau^2} \\ = -3\{(h_{30}\sin\theta - h_{40}\cos\theta)z + \tfrac{1}{2}a_{11}\bar{\omega}\}\,\frac{d\mathfrak{V}}{d\tau} . \end{aligned}\right\} \tag{863}$$

Equations (860) and (863) are consistent only if

$$a_{10} = c_{10} . \tag{864}$$

The solution for the coefficients of the velocity ellipsoid for the case (862) can now be written down. We have

$$\left.\begin{aligned}
a &= (h_{52}\sin 2\theta - h_{62}\cos 2\theta)z^2 + a_{10} + a_{11}z + a_{12}z^2 ,\\
b &= -2\{(h_{30}+h_{31}z)\sin\theta - (h_{40}+h_{41}z)\cos\theta\}\bar\omega\\
&\quad - (h_{52}\sin 2\theta - h_{62}\cos 2\theta)z^2 + a_{10} + a_{11}z + a_{12}z^2 + b_{20}\bar\omega^2 ,\\
c &= (h_{52}\sin 2\theta - h_{62}\cos 2\theta + a_{12})\bar\omega^2\\
&\qquad - 2(h_{30}\sin\theta - h_{40}\cos\theta)\bar\omega + a_{10} ,\\
f &= -\{(h_{52}\cos 2\theta + h_{62}\sin 2\theta)\bar\omega - (h_{30}\cos\theta + h_{40}\sin\theta)\}z\\
&\qquad - (h_{31}\cos\theta + h_{41}\sin\theta)\bar\omega^2 ,\\
g &= \{(-h_{52}\sin 2\theta + h_{62}\cos 2\theta + a_{12})z - \tfrac{1}{2}a_{11}\}\bar\omega\\
&\qquad + (h_{30}\sin\theta - h_{40}\cos\theta)z ,\\
h &= (h_{30}+h_{31}z)\cos\theta + (h_{40}+h_{41}z)\sin\theta\\
&\qquad + (h_{52}\cos 2\theta + h_{62}\sin 2\theta)z^2 .
\end{aligned}\right\} \qquad (865)$$

Finally, considering the motions of the local centroids, we have (cf. Eq. [732])

$$\left.\begin{aligned}
\Delta_1 &= (\alpha_1\cos\theta + \alpha_2\sin\theta)z ,\\
\Delta_2 &= (-\alpha_1\sin\theta + \alpha_2\cos\theta)z + p\bar\omega ,\\
\Delta_3 &= -(\alpha_1\cos\theta + \alpha_2\sin\theta)\bar\omega ,
\end{aligned}\right\} \qquad (866)$$

where the Δ's are related to the components Π_0, Θ_0, and Z_0 of the motion of the local centroid by the relation (42).

VIII. ELLIPSOIDAL SYSTEMS

36. *The fundamental equations for ellipsoidal systems.*—In §§ 34 and 35 we have considered the case where the equipotential surfaces are concentric spheroids. We shall now discuss more generally the case where the equipotential surfaces are concentric ellipsoids. In other words, we shall assume that

$$\mathfrak{B}(x, y, z) \equiv \mathfrak{B}(\tau) , \qquad (867)$$

where

$$\tau = \tfrac{1}{2}(\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2) , \qquad (868)$$

where α_1, α_2, and α_3 are three positive real numbers. If $\alpha_1 \neq \alpha_2 \neq \alpha_3$, then from the theorem proved in Part IV concerning the helical symmetry of stellar systems with differential motions it follows that

$$\Delta_1 = \Delta_2 = \Delta_3 = 0 \qquad (\alpha_1 \neq \alpha_2 \neq \alpha_3) . \quad (869)$$

On the other hand, if two of the α's are equal, then we have the case of spheroidal systems already considered. Finally, if $\alpha_1 = \alpha_2 = \alpha_3$, then the problem reduces to the case of spherical symmetry. We may notice that there will be no loss of generality if we set one of the α's (say, α_1) equal to unity. But the problem can be treated more symmetrically if we retain all the α's.

As we should expect, the discussion is most conveniently carried out in Cartesian co-ordinates. The problem now hinges on the relations (IV_4)

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \operatorname{grad} \mathfrak{B} = -\tfrac{1}{2} \operatorname{grad} \chi . \quad (870)$$

The integrability conditions are obtained by taking the curl of the foregoing equation when the right-hand side vanishes. We obtain in this way

$$\left.\begin{aligned}
3\frac{\partial h}{\partial x}\frac{\partial \mathfrak{B}}{\partial x} - 3\frac{\partial h}{\partial y}\frac{\partial \mathfrak{B}}{\partial y} + \left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right)\frac{\partial \mathfrak{B}}{\partial z} + h\left(\frac{\partial^2 \mathfrak{B}}{\partial x^2} - \frac{\partial^2 \mathfrak{B}}{\partial y^2}\right)& \\
+ (b-a)\frac{\partial^2 \mathfrak{B}}{\partial x \partial y} - g\frac{\partial^2 \mathfrak{B}}{\partial y \partial z} + f\frac{\partial^2 \mathfrak{B}}{\partial z \partial x} = 0 ,& \\
3\frac{\partial f}{\partial y}\frac{\partial \mathfrak{B}}{\partial y} - 3\frac{\partial f}{\partial z}\frac{\partial \mathfrak{B}}{\partial z} + \left(\frac{\partial g}{\partial y} - \frac{\partial h}{\partial z}\right)\frac{\partial \mathfrak{B}}{\partial x} + f\left(\frac{\partial^2 \mathfrak{B}}{\partial y^2} - \frac{\partial^2 \mathfrak{B}}{\partial z^2}\right)& \\
+ (c-b)\frac{\partial^2 \mathfrak{B}}{\partial y \partial z} - h\frac{\partial^2 \mathfrak{B}}{\partial z \partial x} + g\frac{\partial^2 \mathfrak{B}}{\partial x \partial y} = 0 ,& \\
3\frac{\partial g}{\partial z}\frac{\partial \mathfrak{B}}{\partial z} - 3\frac{\partial g}{\partial x}\frac{\partial \mathfrak{B}}{\partial x} + \left(\frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}\right)\frac{\partial \mathfrak{B}}{\partial y} + g\left(\frac{\partial^2 \mathfrak{B}}{\partial z^2} - \frac{\partial^2 \mathfrak{B}}{\partial x^2}\right)& \\
+ (a-c)\frac{\partial^2 \mathfrak{B}}{\partial z \partial x} - f\frac{\partial^2 \mathfrak{B}}{\partial x \partial y} + h\frac{\partial^2 \mathfrak{B}}{\partial y \partial z} = 0 .&
\end{aligned}\right\} \quad (871)$$

When $\mathfrak{V}$ has the form (867), we have

$$\left.\begin{aligned}
\frac{\partial\mathfrak{V}}{\partial x} &= \alpha_1 x\frac{d\mathfrak{V}}{d\tau}\,; & \frac{\partial^2\mathfrak{V}}{\partial x\partial y} &= \alpha_1\alpha_2 xy\frac{d^2\mathfrak{V}}{d\tau^2}\,,\\
\frac{\partial\mathfrak{V}}{\partial y} &= \alpha_2 y\frac{d\mathfrak{V}}{d\tau}\,; & \frac{\partial^2\mathfrak{V}}{\partial y\partial z} &= \alpha_2\alpha_3 yz\frac{d^2\mathfrak{V}}{d\tau^2}\,,\\
\frac{\partial\mathfrak{V}}{\partial z} &= \alpha_3 z\frac{d\mathfrak{V}}{d\tau}\,; & \frac{\partial^2\mathfrak{V}}{\partial z\partial x} &= \alpha_3\alpha_1 zx\frac{d^2\mathfrak{V}}{d\tau^2}\,,
\end{aligned}\right\} \tag{872}$$

and

$$\left.\begin{aligned}
\frac{\partial^2\mathfrak{V}}{\partial x^2} &= \alpha_1^2x^2\frac{d^2\mathfrak{V}}{d\tau^2}+\alpha_1\frac{d\mathfrak{V}}{d\tau}\,,\\
\frac{\partial^2\mathfrak{V}}{\partial y^2} &= \alpha_2^2y^2\frac{d^2\mathfrak{V}}{d\tau^2}+\alpha_2\frac{d\mathfrak{V}}{d\tau}\,,\\
\frac{\partial^2\mathfrak{V}}{\partial z^2} &= \alpha_3^2z^2\frac{d^2\mathfrak{V}}{d\tau^2}+\alpha_3\frac{d\mathfrak{V}}{d\tau}\,.
\end{aligned}\right\} \tag{873}$$

Using the foregoing relations in the equations (871), we find that these integrability conditions reduce to

$$\left.\begin{aligned}
&\left\{3\alpha_1 x\frac{\partial h}{\partial x}-3\alpha_2 y\frac{\partial h}{\partial y}+\alpha_3 z\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right)+(\alpha_1-\alpha_2)h\right\}\frac{d\mathfrak{V}}{d\tau}\\
&\quad=-\{h(\alpha_1^2x^2-\alpha_2^2y^2)+\alpha_1\alpha_2 xy(b-a)+\alpha_3\alpha_1 zxf-\alpha_2\alpha_3 yzg\}\frac{d^2\mathfrak{V}}{d\tau^2}\,,\\
&\left\{3\alpha_2 y\frac{\partial f}{\partial y}-3\alpha_3 z\frac{\partial f}{\partial z}+\alpha_1 x\left(\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}\right)+(\alpha_2-\alpha_3)f\right\}\frac{d\mathfrak{V}}{d\tau}\\
&\quad=-\{f(\alpha_2^2y^2-\alpha_3^2z^2)+\alpha_2\alpha_3 yz(c-b)+\alpha_1\alpha_2 xyg-\alpha_3\alpha_1 zxh\}\frac{d^2\mathfrak{V}}{d\tau^2}\,,\\
&\left\{3\alpha_3 z\frac{\partial g}{\partial z}-3\alpha_1 x\frac{\partial g}{\partial x}+\alpha_2 y\left(\frac{\partial h}{\partial z}-\frac{\partial f}{\partial x}\right)+(\alpha_3-\alpha_1)g\right\}\frac{d\mathfrak{V}}{d\tau}\\
&\quad=-\{g(\alpha_3^2z^2-\alpha_1^2x^2)+\alpha_3\alpha_1 zx(a-c)+\alpha_2\alpha_3 yzh-\alpha_1\alpha_2 xyf\}\frac{d^2\mathfrak{V}}{d\tau^2}\,.
\end{aligned}\right\} \tag{874}$$

37. *Motions inside a homogeneous ellipsoidal distribution of mass.*—As is well known, the equipotential surfaces inside a homogeneous ellipsoidal distribution of mass are concentric ellipsoids. Further, we have in this case

$$\mathfrak{V} = \text{constant} + \text{constant}\,(\alpha_1x^2+\alpha_2y^2+\alpha_3z^2)\,, \tag{875}$$

where α_1, α_2, and α_3 will depend on the density and the ratio of the axes of the homogeneous ellipsoid. For the case (875)

$$\frac{d^2\mathfrak{B}}{d\tau^2} = 0 \,. \tag{876}$$

Hence, for this case the validity of the equations (874) requires that the terms which occur as the coefficients of $d\mathfrak{B}/d\tau$ on the left-hand sides of these equations should all vanish. We should therefore have

$$\left.\begin{aligned}
3\alpha_1 x\,\frac{\partial h}{\partial x} - 3\alpha_2 y\,\frac{\partial h}{\partial y} + \alpha_3 z\left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right) + (\alpha_1 - \alpha_2)h &= 0\,,\\
3\alpha_2 y\,\frac{\partial f}{\partial y} - 3\alpha_3 z\,\frac{\partial f}{\partial z} + \alpha_1 x\left(\frac{\partial g}{\partial y} - \frac{\partial h}{\partial z}\right) + (\alpha_2 - \alpha_3)f &= 0\,,\\
3\alpha_3 z\,\frac{\partial g}{\partial z} - 3\alpha_1 x\,\frac{\partial g}{\partial x} + \alpha_2 y\left(\frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}\right) + (\alpha_3 - \alpha_1)g &= 0\,.
\end{aligned}\right\} \tag{877}$$

Substituting in the foregoing equations for f, g, and h according to the general solution obtained in Part III (Eqs. [183]–[185]) and equating the coefficients of the different powers and the different combinations of the powers of x, y, and z, we find

$$\left.\begin{aligned}
&(\alpha_1 - \alpha_2)h_{10} = 0\,; && (\alpha_2 - \alpha_3)f_{10} = 0\,; && (\alpha_3 - \alpha_1)g_{10} = 0\,,\\
&(4\alpha_1 - \alpha_2)h_{20} = 0\,; && (4\alpha_2 - \alpha_3)f_{20} = 0\,; && (4\alpha_3 - \alpha_1)g_{20} = 0\,,\\
&(\alpha_1 - 4\alpha_2)h_{30} = 0\,; && (\alpha_2 - 4\alpha_3)f_{30} = 0\,; && (\alpha_3 - 4\alpha_1)g_{30} = 0\,,\\
&\alpha_3(f_{11}-g_{11})+(\alpha_1-\alpha_2)h_{11}=0; && \alpha_1(g_{11}-h_{11})+(\alpha_2-\alpha_3)f_{11}=0; && \alpha_2(h_{11}-f_{11})+(\alpha_3-\alpha_1)g_{11}=0,\\
&(\alpha_1 - \alpha_2)h_{40} = 0\,; && (\alpha_2 - \alpha_3)f_{40} = 0\,; && (\alpha_3 - \alpha_1)g_{40} = 0\,,\\
&(\alpha_1 - 4\alpha_2 + 3\alpha_3)f_{21} = 0\,; && (\alpha_2 - 4\alpha_3 + 3\alpha_1)g_{21} = 0\,; && (\alpha_3 - 4\alpha_1 + 3\alpha_2)h_{21} = 0\,,\\
&(4\alpha_1 - \alpha_2 - 3\alpha_3)h_{21} = 0\,; && (4\alpha_2 - \alpha_3 - 3\alpha_1)f_{21} = 0\,; && (4\alpha_3 - \alpha_1 - 3\alpha_2)g_{21} = 0\,,\\
&(\alpha_1 - \alpha_2)g_{21} = 0\,; && (\alpha_2 - \alpha_3)h_{21} = 0\,; && (\alpha_3 - \alpha_1)f_{21} = 0\,.
\end{aligned}\right\} \tag{878}$$

Case I: $\alpha_1 \neq \alpha_2 \neq \alpha_3$.—We shall set $\alpha_1 = 1$; as we have already remarked, this implies no loss of generality. Our assumption is therefore equivalent to

$$\alpha_1 = 1 \neq \alpha_2 \neq \alpha_3 \,. \tag{879}$$

From the equations (878) we now obtain

$$f_{10} = g_{10} = h_{10} = f_{21} = g_{21} = h_{21} = f_{40} = g_{40} = h_{40} = 0 \,. \tag{880}$$

Further,

$$\left.\begin{array}{lll} (4 - a_2)h_{20} = 0 \,; & (4a_2 - a_3)f_{20} = 0 \,; & (4a_3 - 1)g_{20} = 0 \,, \\ (1 - 4a_2)h_{30} = 0 \,; & (a_2 - 4a_3)f_{30} = 0 \,; & (a_3 - 4)g_{30} = 0 \,, \end{array}\right\} \tag{881}$$

and

$$\left.\begin{array}{rrr} a_3 f_{11} - & a_3 g_{11} + (1 - a_2)h_{11} & = 0 \,, \\ (a_2 - a_3)f_{11} + & g_{11} - \quad h_{11} & = 0 \,, \\ -a_2 f_{11} + (a_3 - 1)g_{11} + & a_2 h_{11} & = 0 \,. \end{array}\right\} \tag{882}$$

The three equations (882) should be further supplemented by the condition (cf. Eq. [186])

$$f_{11} + g_{11} + h_{11} = 0 \,. \tag{883}$$

Hence, if f_{11}, g_{11}, and h_{11} are not all zero, the four three-rowed determinants of the matrix

$$\begin{pmatrix} a_3 & -a_3 & 1 - a_2 \\ a_2 - a_3 & 1 & -1 \\ -a_2 & a_3 - 1 & a_2 \\ 1 & 1 & 1 \end{pmatrix} \tag{884}$$

should all vanish. The determinant formed by the first three rows of the matrix (884) vanishes identically, and the condition for the vanishing of any of the other three determinants of order 3 of the matrix (884) is found to be

$$1 + a_2^2 + a_3^2 = 2(a_2 + a_3 + a_2 a_3) \,; \tag{885}$$

or, solving for a_3 in terms of a_2, we find

$$\sqrt{a_3} - \sqrt{a_2} = \pm 1 \,. \tag{886}$$

If the relation (886) is not satisfied, then f_{11}, g_{11}, and h_{11} all vanish. On the other hand, if the relation (886) is valid, then

$$\frac{f_{11}}{\alpha_3 - \alpha_2 + 1} = \frac{g_{11}}{\alpha_3 + \alpha_2 - 1} = \frac{h_{11}}{-2\alpha_3} = \text{constant} . \tag{887}$$

We return now to the relations (881). We readily verify that a discussion of these relations leads to the following four physically distinct cases:

$$\left.\begin{aligned}
&\alpha_i : \alpha_j : \alpha_k = 1 : 4 : 16 && (\text{case I}_a) , \\
&\alpha_i : \alpha_j : \alpha_k = 1 : 4 : 9 && (\text{case I}_b) , \\
&\alpha_i : \alpha_j : \alpha_k = 1 : 4 : \alpha_j \ (\neq 16, 9, \tfrac{1}{4}) && (\text{case I}_c) , \\
&\alpha_i : \alpha_j : \alpha_k \neq 1 : 4 : 16 && (\text{case I}_d) .
\end{aligned}\right\} \tag{888}$$

As illustrative of these four cases we shall consider the following typical examples.

Case I_a: $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = 16$.—From the relations (881) we now conclude that

$$h_{30} = f_{30} = g_{30} = g_{20} = 0 \ ; \qquad h_{20}, f_{20} \neq 0 . \tag{889}$$

Also, since equation (886) is not satisfied for this choice of the α's,

$$f_{11} = g_{11} = h_{11} = 0 . \tag{890}$$

From the general solution for the coefficients of the velocity ellipsoid obtained in Part III (cf. Eqs. [183]–[185] and [191]–[193]) we find that the solution appropriate for the present case is given by

$$\left.\begin{aligned}
a = -2h_{20}y - a_0 \ ; \qquad b = -2f_{20}z - b_0 \ ; \qquad c = -c_0 \ , \\
f = f_{20}y \ ; \qquad g = 0 \ ; \qquad h = h_{20}x \ .
\end{aligned}\right\} \tag{891}$$

The solution contains five arbitrary constants.

Case I_b: $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = 9$.—From the relations (881) we now infer that

$$f_{30} = g_{30} = h_{30} = f_{20} = g_{20} = 0; \qquad h_{20} \neq 0 . \tag{892}$$

On the other hand, since this choice of the α's satisfies the relation (886), we have (according to Eq. [887])

$$f_{11} = \kappa \; ; \qquad g_{11} = 2\kappa \; ; \qquad h_{11} = -3\kappa \, , \tag{893}$$

where κ is an arbitrary constant. The solution for a, b, c, f, g, and h for the present case can now be written down from the general solution. We have

$$\left.\begin{aligned} a &= -2h_{20}y - a_0 \; ; \qquad b = -b_0 \; ; \qquad c = -c_0 \, , \\ f &= \kappa x \; ; \qquad g = 2\kappa y \; ; \qquad h = -3\kappa z + h_{20}x \, . \end{aligned}\right\} \tag{894}$$

This solution also involves five arbitrary constants.

Case I_c: $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 \neq 16, 9, \frac{1}{4}$.—For this case we readily find that

$$\left.\begin{aligned} a &= -2h_{20}y - a_0 \; ; \qquad b = -b_0 \; ; \qquad c = -c_0 \, , \\ f &= 0 \; ; \qquad g = 0 \; ; \qquad h = h_{20}x \, . \end{aligned}\right\} \tag{895}$$

Case I_d: $\alpha_i : \alpha_j : \alpha_k \neq 1:4:16$.—From the relations (881) we now conclude that

$$h_{20} = f_{20} = g_{20} = h_{30} = f_{30} = g_{30} = 0 \, . \tag{896}$$

Hence,

$$a = -a_0 \; ; \qquad b = -b_0 \; ; \qquad c = -c_0 \, . \tag{897}$$

If, further, equation (886) is not satisfied, then

$$f = g = h = 0 \qquad (\sqrt{\alpha_2} \neq \sqrt{\alpha_3} \pm 1) \, . \tag{898}$$

On the other hand, if (886) is satisfied, then

$$f = f_{11}x \; ; \qquad g = g_{11}y \; ; \qquad h = h_{11}z \, , \tag{899}$$

where f_{11}, g_{11}, and h_{11} are expressed in terms of a single arbitrary constant according to equation (887).

Turning next to the differential motions, it is clear that for the case under consideration ($\alpha_1 \neq \alpha_2 \neq \alpha_3$) the Δ's vanish (cf. Eq.

[869]). Hence, if the components of the motions of the local centroids do not all vanish identically, then the fundamental determinant (275) should vanish. If we substitute for the coefficients of the velocity ellipsoid the expressions found for the four cases considered above, we find that no really significant case of differential motions exists.

Case II: $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = \alpha \neq 1$.—From the equations (878) we find

$$f_{10} = g_{10} = h_{20} = h_{30} = f_{21} = g_{21} = h_{21} = f_{40} = g_{40} = 0 \qquad (900)$$

and

$$h_{10},\, h_{40} \neq 0\,. \qquad (901)$$

Further, we have

$$\left.\begin{aligned} (4-\alpha)f_{20} &= 0\,; \qquad (1-4\alpha)g_{20} = 0\,, \\ (4-\alpha)g_{30} &= 0\,; \qquad (1-4\alpha)f_{30} = 0\,, \end{aligned}\right\} \qquad (902)$$

and

$$f_{11} = g_{11}\,; \qquad g_{11} - h_{11} + (1-\alpha)f_{11} = 0\,. \qquad (903)$$

Equation (903) should be supplemented by

$$f_{11} + g_{11} + h_{11} = 0\,. \qquad (904)$$

Equations (903) and (904) can be combined to give

$$f_{11} = g_{11} = -\tfrac{1}{2}h_{11}\,; \qquad (4-\alpha)f_{11} = 0\,. \qquad (905)$$

As in § 34, we have to distinguish between the following three cases.

Case II_a: $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = \alpha \neq 1, 4, \tfrac{1}{4}$.—In addition to (900) we have

$$f_{11} = g_{11} = h_{11} = f_{20} = g_{20} = f_{30} = g_{30} = 0\,. \qquad (906)$$

The solution for the coefficients of the velocity ellipsoid is seen to be

$$\left.\begin{aligned} a = -a_0 - h_{40}y^2\,; \qquad b = -b_0 - h_{40}x^2\,; \qquad c = -c_0\,, \\ f = g = 0\,; \qquad h = h_{10} + h_{40}xy\,. \end{aligned}\right\} \qquad (907)$$

Case II_b: $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 4$.—For this case

$$f_{20},\, g_{30} \neq 0; \qquad g_{20} = f_{30} = 0\,. \tag{908}$$

Further,

$$f_{11} = g_{11} = -\tfrac{1}{2}h_{11} = \kappa\,, \tag{909}$$

where κ is an arbitrary constant. We now have

$$\left.\begin{aligned} a &= -a_0 - 2g_{30}z - h_{40}y^2\,, \\ b &= -b_0 - 2f_{20}z - h_{40}x^2\,, \\ c &= -c_0\,, \\ f &= \kappa x + f_{20}y\,; \qquad g = \kappa y + g_{30}x\,, \\ h &= h_{10} - 2\kappa z + h_{40}xy\,. \end{aligned}\right\} \tag{910}$$

Case II_c: $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = \frac{1}{4}$.—From equations (902) and (905) we infer that

$$g_{20},\, f_{30} \neq 0\,; \qquad f_{11} = g_{11} = h_{11} = f_{20} = g_{30} = 0\,, \tag{911}$$

and we have

$$\left.\begin{aligned} a &= -a_0 - h_{40}y^2\,; \qquad b = -b_0 - h_{40}x^2\,, \\ c &= -c_0 - 2g_{20}x - 2f_{30}y\,, \\ f &= f_{30}z\,; \qquad g = g_{20}z\,, \\ h &= h_{10} + h_{40}xy\,. \end{aligned}\right\} \tag{911$'$}$$

Turning next to a consideration of the differential motions, since for the case under consideration the systems have simple axial symmetries about the z-axis, we have (cf. Eqs. [205] and [269])

$$\Delta_1 = \beta y\,; \qquad \Delta_2 = -\beta x\,; \qquad \Delta_3 = 0\,, \tag{912}$$

where β is a constant. The components U_0, V_0, and W_0 of the motion of the local centroid can now be obtained from the appropriate solutions for the coefficients of the velocity ellipsoid.

Case III: $\alpha_1 = \alpha_2 = \alpha_3$.—When $\alpha_1 = \alpha_2 = \alpha_3$, the relations (878) give

$$f_{20} = g_{20} = h_{20} = f_{30} = g_{30} = h_{30} = f_{11} = g_{11} = h_{11} = 0\,. \tag{913}$$

On the other hand,

$$f_{10},\ g_{10},\ h_{10},\ f_{21},\ g_{21},\ h_{21},\ f_{40},\ g_{40},\ h_{40} \neq 0\ . \tag{914}$$

The appropriate solution for the coefficients of the velocity ellipsoid can now be written down from our general solution and is seen to involve twelve arbitrary constants.

38. *The general discussion of the relations* (874).—If the α's are not all equal and if, further, $d^2\mathfrak{B}/d\tau^2 \neq 0$, it can then be readily shown that the integrability conditions must be satisfied identically. In other words, the terms which occur as the coefficients of $d^2\mathfrak{B}/d\tau^2$ and of $d\mathfrak{B}/d\tau$ in the relations (874) must all vanish. By writing down the conditions for the vanishing of these terms, we find that

$$a = b = c = \text{constant}\ ; \qquad f = g = h = 0 \qquad (\alpha_1 \neq \alpha_2 \neq \alpha_3) \tag{915}$$

and

$$\left.\begin{aligned} a &= -a_0 - h_{40}y^2\ ; \qquad b = -a_0 - h_{40}x^2\ , \\ c &= -a_0\ ; \qquad h = h_{40}xy\ ; \qquad f = g = 0\ . \end{aligned} \quad \begin{pmatrix} \alpha_1 = \alpha_2 = 1, \\ \alpha_3 \neq 1 \end{pmatrix}\right\} \tag{916}$$

Equation (916) is in agreement with the results obtained in § 34.

The general discussion of the integrability relations (874) for the case of spherical symmetry ($\alpha_1 = \alpha_2 = \alpha_3$) has some features of interest.

Substituting in (874) (in which α_1, α_2, and α_3 have all been set equal to unity) for the coefficients of the velocity ellipsoid according to the general solution obtained in Part III, we find

$$\left.\begin{aligned} &\{(h_{20}x - h_{30}y)(x^2 + y^2) + (f_{30}x - g_{20}y)z^2 + (h_{11} + f_{11})x^2z - (h_{11} + g_{11})y^2z \\ &\quad + (g_{30} - f_{20})xyz - g_{10}yz + f_{10}zx + h_{10}(x^2 - y^2) + (a_0 - b_0)xy\}\, \frac{d^2\mathfrak{B}}{d\tau^2} \\ &= -\{3(h_{20}x - h_{30}y) + (f_{11} - g_{11})z\}\, \frac{d\mathfrak{B}}{d\tau}\ , \end{aligned}\right\} \tag{917}$$

and two similar equations.

If equation (917) (and the two other similar equations) should be satisfied identically, then

$$\left.\begin{aligned} h_{20} = h_{30} = f_{20} = f_{30} = g_{20} = g_{30} = h_{10} = f_{10} &= g_{10} \\ = h_{11} = f_{11} = g_{11} &= 0 \end{aligned}\right\} \tag{918}$$

and

$$a_0 = b_0 = c_0 = -\kappa \text{ (say)} . \tag{919}$$

The solution for the coefficients of the velocity ellipsoid is given by

$$\left.\begin{aligned} a &= \kappa - h_{40}y^2 - g_{40}z^2 - 2h_{21}yz , \\ b &= \kappa - f_{40}z^2 - h_{40}x^2 - 2f_{21}zx , \\ c &= \kappa - g_{40}x^2 - f_{40}y^2 - 2g_{21}xy , \\ f &= -h_{21}x^2 + f_{21}xy + g_{21}xz + f_{40}yz , \\ g &= -f_{21}y^2 + g_{21}yz + h_{21}yx + g_{40}zx , \\ h &= -g_{21}z^2 + h_{21}zx + f_{21}zy + h_{40}xy . \end{aligned}\right\} \tag{920}$$

If equation (917) is not satisfied identically, then the ratio of the coefficients of $d^2\mathfrak{V}/d\tau^2$ and $d\mathfrak{V}/d\tau$ in this equation must be a function of τ only. This can be arranged in only one of the following two ways: either

$$h_{20} = h_{30} = f_{20} = f_{30} = g_{20} = g_{30} = f_{11} = g_{11} = h_{11} = 0 , \tag{921}$$

in which case

$$\frac{d^2\mathfrak{V}}{d\tau^2} = 0 , \tag{922}$$

or

$$\left.\begin{aligned} & h_{20} = f_{30} ; \qquad f_{20} = g_{30} ; \qquad g_{20} = h_{30} , \\ & f_{10} = g_{10} = h_{10} = f_{11} = g_{11} = h_{11} = 0 , \\ & a_0 = b_0 = c_0 , \end{aligned}\right\} \tag{923}$$

in which case

$$\frac{d^2\mathfrak{V}}{d\tau^2} = -\frac{3}{x^2 + y^2 + z^2}\frac{d\mathfrak{V}}{d\tau} = -\frac{3}{2\tau}\frac{d\mathfrak{V}}{d\tau} . \tag{924}$$

The solutions appropriate for these two cases can now be written down from our general solution.

YERKES OBSERVATORY
March 1939

THE ARC SPECTRUM OF EUROPIUM*

HENRY NORRIS RUSSELL AND ARTHUR S. KING

ABSTRACT

Most of the strong lines of *Eu* I arise from transitions between terms of multiplicity 10, 8, and 6, obtained by adding two electrons to $(4f)^7$ $^8S^o$ of *Eu* III. This part of the spectrum is regular in every respect and contains typical multiplets and numerous short series.

The *lowest level* is f^7s^2 $^8S^o$. All the terms arising from the added electrons 6s7s, 6s8s, 6s5d, 6s6d, 6s6p, 6s7p, and 5d6p have been identified, and some from 6s7d, 6s5f, 6s6f, $(5d)^2$, and $(6p)^2$. There are many *additional energy-levels*, which probably arise from the configurations f^6ds^2 and f^6d^2s.

The *number of classified lines* is 1156, including all but one of the 200 strongest.

The *ionization potential is 5.64 volts.*

Tables are given showing the *terms and unclassified levels*, the *electron configurations*, the *series relations*, the *intensities* in *multiplets*, and a *complete list of classified lines*.

The *probable error* of a tabular wave length is ± 0.010 A for the stronger lines and ± 0.017 for the faintest group.

I. INTRODUCTION

Europium is a typical example of those rare earths whose spectra show a moderate number of very strong lines and are therefore more promising for analysis.[1] The principal features of the first spark spectrum have been interpreted by Albertson,[2] who is continuing the study of the numerous lines not yet classified. A preliminary report on the arc spectrum has been made by the writers.[3] The analysis here reported is based upon new observations by one of us, covering the range from 2100 to 10100 A, which are described in detail in another communication.[4] The kindness of Dr. H. N. McCoy in providing a generous supply of the rare material has led to the discovery of hundreds of faint lines, which have proved of great importance in extending the analysis. Observations of the Zeeman effect were impracticable, mainly because it is very difficult, with an element of

* *Contributions from the Mount Wilson Observatory, Carnegie Institute of Washington*, No. 611.

[1] W. Albertson, *Phys. Rev.*, **47**, 370, 1935.

[2] *Ibid.*, **45**, 499, 1934.

[3] *Phys. Rev.*, **46**, 1023, 1934.

[4] King, *Mt. Wilson Contr.*, No. 608; *Ap. J.*, **89**, 377, 1939.

such low ionization potential, to get anything but spark lines in the magnetic field. This analysis has been carried far enough to permit the classification of practically all the stronger arc lines and a majority of the weaker ones, and has revealed a very regular structure, in excellent agreement with Hund's theory. Since the type of structure differs considerably from any which is found among the lighter elements, it is described here in more detail than would otherwise be the case.

2. GENERAL STRUCTURE OF THE SPECTRUM

The neutral atom of europium contains nine electrons outside the closed "krypton shell." The very existence of a moderate number of dominant strong lines in both the arc and the spark spectra proves that seven of these are 4f-electrons. The electron group f^7 gives rise, theoretically, to the spectroscopic terms $^8S^o$, $^6P^o$, $^6D^o$, $^6F^o$, $^6G^o$, $^6H^o$, and $^6I^o$ and to very numerous quartets and doublets. By analogy with the half-shells d^5 and p^3, we should expect the $^8S^o$ term to lie much lower than the rest. By far the greater part of the whole energy of the arc spectrum arises from transitions between states obtained by adding two other electrons to this one term of *Eu* III (though there is evidence also of states involving six f-electrons).

In the main, therefore, *Eu* I is a two-electron spectrum; but the high multiplicity of the limit makes its structure richer than that of other two-electron spectra. The two electrons may be in the states 6s, 6p, 5d, and 5f or have higher total quantum numbers.

By analogy with *Ba* I, which differs only in the absence of the f^7 group, we should expect the lowest state to involve $(6s)^2$, and the next lowest 6s5d. These should combine strongly with higher states arising from 6s6p and 5d6p. In the present case the lowest terms are of odd parity, and the higher ones even.

From $(^8S^o)(6s)^2$ we have only $^8S^o_{3\frac{1}{2}}$, while $(^8S^o)$6s6p gives ^{10}P, ^8P, ^8P, and ^6P. The first of these should be the lowest, and its combination with the ground state should give two strong low-temperature lines in the red, while the transition from these levels to the higher 6s7s $^{10}S^o$ should give another pair in the same region and with the same frequency separation.

At a first glance at the plates, two pairs of lines of the anticipated

character were conspicuous. Their wave numbers, intensities, and temperature classes (from the final list) are:

	14563.61 (3000 I)	14200.24 (2500 II)
	14067.79 (1500 I)	14695.95 (2500 II)
Diff...	495.82	495.71

The first "break" into the analysis was thus obtained by inspection. The remaining component $^{10}P_{5\frac{1}{2}}$ and the two ^{8}P terms were then easily found. Two higher $^{10}S^{o}$ and two $^{10}D^{o}$ terms, combining with these, gave series and an ionization potential of 5.64 volts.[3] The ^{6}P term was found later.

The configuration $(^{8}S^{o})6s5d$ should give $^{10}D^{o}$, $^{8}D^{o}$, $^{8}D^{o}$, and $^{6}D^{o}$, the first two going to the limit $(^{8}S^{o})5d$ $^{9}D^{o}$ of *Eu* II, and the others to the related $^{7}D^{o}$ term.[5] These terms should combine with four triads of PDF terms, of corresponding multiplicities, giving strong lines in the red, yellow, and green. A great tangle of such lines is present. To unravel it, use was made of the ultraviolet lines, which show strong absorption in the furnace. All such lines may confidently be attributed to transitions from the ground state $^{8}S^{o}$, and, since this is single, each one locates a new energy-level.

By "combing" the list in the red with the differences of these levels, transitions downward were found which located the $^{10}D^{o}$ and the lower $^{8}D^{o}$ terms and then a large number of higher even levels. The term $b^{8}D^{o}$ was found laboriously by comparing the differences of these levels with those among the fifty strongest remaining lines in the yellow. Had the infrared then been available, it could have been found at once from the isolated multiplet $b^{8}D^{o}-z^{8}D$. The $^{6}D^{o}$ term was finally located by means of infrared combinations. These $^{8}D^{o}$ and $^{6}D^{o}$ terms are inverted like the parent $^{7}D^{o}$ term of *Eu* II.

The combinations of these terms practically exhaust the strongest lines throughout the spectrum, making it evident that no other low terms exist. The whole number of lines classified is 1156. All but five of these arise from combinations between the 21 odd and 12

[5] The latter (which was revealed by a multiplet near λ 10000, discovered in the course of the present observations) lies about 7500 cm^{-1} above $a^{9}D^{o}$ and is inverted, while the former has its levels in normal order.

even levels already described, and 78 odd and 161 even higher levels. A complete list of these is given in section 7.

3. TERM AND CONFIGURATION ANALYSIS

The grouping of most of these levels into terms was easy, as there are many multiplets, remarkably regular in structure and intensity; and the assignment of electron configurations was usually simple.

The terms theoretically produced by the addition of two electrons to $(4f)^7\ {}^8S^o$ of *Eu* III are listed in Table 1. Terms predicted but not

TABLE 1

TERMS AND ELECTRON CONFIGURATIONS IN *Eu* I

Configurations	Terms			
$(6s)^2$	. . . .	a^8S^o		. . . .
6s7s	$e^{10}S^o$	e^8S^o	f^8S^o	e^6S^o
6s8s	$f^{10}S^o$	g^8S^o	i^8S^o	f^6S^o
6s5d	$a^{10}D^o$	a^8D^o	b^8D^o	a^6D^o
6s6d	$e^{10}D^o$	e^8D^o	g^8D^o	e^6D^o
6s7d	$f^{10}D^o$	h^8D^o	$({}^8D^o)$	f^6D^o
$(5d)^2$	$e^{10}P^o$, $({}^{10}F^o)$	h^8S^o, e^8P^o, f^8D^o, $({}^8F^o)$, $({}^8G^o)$		e^6P^o, $({}^6F^o)$
$(6p)^2$	$f^{10}P^o$	$({}^8S^o)$, $({}^8P^o)$, $({}^8D^o)$		$({}^6P^o)$
6s6p	$z^{10}P$	z^8P	y^8P	z^6P
6s7p	$x^{10}P$	w^8P	u^8P	x^6P?
6s5f	$y^{10}F$	$({}^8F)$	x^8F	y^6F
6s6f	$x^{10}F$	$({}^8F)$	$({}^8F)$	$({}^6F)$
5d6p	$y^{10}P$	x^8P	v^8P	y^6P
. . . .	$z^{10}D$	z^8D	y^8D	z^6D
. . . .	$z^{10}F$	z^8F	y^8F	z^6F

identified are in parentheses. The terms arising from $(6s)^2$ 6s7s, 6s5d, 6s6d, 6s6p, and 5d6p may thus be determined with certainty. Those involving 8s and 7d are identified by series relations (sec. 4).

The existence of two ${}^{10}P^o$ terms is clearly shown by the multiplets which they form. Such terms can arise by the addition to the ${}^8S^o$ limit, of two electrons which, by themselves, would give a ${}^3P$ term—that is, $(5d)^2$ or $(6p)^2$. The lower of the two has been assigned to d^2 because of its narrower separations, and since the corresponding term in *Ba* I is the lower. The ${}^6P^o$ term from $(5d)^2$ is also unmistakable. There should also be ${}^8S^o$, ${}^8P^o$, and ${}^8D^o$ terms. These lie near the octet terms arising from 6s6d and 6s8s, and the mutual perturbations are

large in position and intensity. The whole number of levels of each J-value is correct, but assignment to individual terms is probably of doubtful significance, owing to "sharing" of their properties. The arrangement finally adopted, though involving some irregularities, appears to be as good as can be made in the absence of Zeeman observations. The F^o and G^o terms arising from this configuration should combine very feebly with the low P terms and have not been identified.

Among the even terms combining with the 6s5d group are four P terms in addition to those belonging to the great triads, which are just what is to be expected from 6s7p. Each of these terms is close to the P term of one of the triads, and there is some evidence of perturbation.

Four additional F terms have been found. All are narrow and are at just the levels where they should be if arising from 6s5f or 6s6f. One of the ^8F terms from 6s5f is missing. It should lie close to y^8F and may be so much perturbed as to be unidentified.

The portion of the spectrum so far discussed contains 50 identified terms, arising from 13 different electron configurations and including 190 component levels. Combinations between these account for 815 observed lines, including 90 per cent of those in the whole spectrum with intensities exceeding 100 (the strongest line being 10,000).

Many additional energy levels have been found—some of them confirmed by six to nine combinations. These are designated by numbers (sec. 6). Those finally admitted include 63 even and 19 odd levels. The limits of admission are difficult to prescribe; in general, a level based on only two combinations is suspicious, unless the lines are strong and of appropriate temperature class. Lines which disagree seriously with the anticipated temperature class have been rejected as accidental coincidences. Three levels have been included on the basis of single lines, strong in furnace absorption, undoubtedly belonging to *Eu* I, and interpretable only as transitions from the ground state. One of these lines, λ 3334.33, is among the strongest in the spectrum.

Combinations including these levels account for 341 lines and leave outstanding only 1 line of intensity 100 or more in the arc among the 231 in the spectrum.

These levels cannot, on any reasonable assumption, be fitted into the scheme based on $(4f)^7\ {}^8S^o$. The even ones, which are much the more important, probably arise from the configurations $(4f)^6(6s)^2 5d$ and $(4f)^6 6s(5d)^2$. The lowest term from $(4f)^6$ should be a normal ${}^7F$, as Albertson[1] has found in samarium. Addition of $(6s)^2 5d$ gives ${}^8PDFGH$, ${}^6PDFGH$, while $6s(5d)^2$ produces ${}^{10}SPDFGHI$, ${}^{10}DFG$ and a great number of octets and sextets. The levels which combine most strongly with $a^{10}D^o$ presumably belong to this configuration. The transition $f^6sd^2 \rightarrow f^7sd$ involves the electron jump $d \rightarrow f$, while $f^6sd^2 \rightarrow f^7s^2$ demands the double jump $d \rightarrow s$, $d \rightarrow f$. The combinations of these levels with a^8S^o should therefore be weak or absent, as is the case. On the other hand, $f^6s^2d \rightarrow f^7s^2$ involves the jump $d \rightarrow f$ and $f^6s^2d \rightarrow f^7sd$, $s \rightarrow d$, $d \rightarrow f$. The levels which give strong combinations with a^8S^o probably belong to this configuration. They lie, on the average, about 8000 cm^{-1} lower than the others. The corresponding difference between the ds^2 and d^2s configurations in *La* I ranges from 2700 to 9900 cm^{-1}.

It is evident that by no means all the levels of these configurations have been detected. Any attempt to pick out multiplets would be unwarranted, especially in the absence of Zeeman data.

The configurations f^8s, f^8d should combine with f^7sd but not with f^7s^2 (as the $s \rightarrow f$ jump is most improbable). One would expect them to lie at a higher level if analogy with groups of d-electrons is valid; whether they are represented in the observations is uncertain.

There remain a good many lines of moderate strength and of temperature classes IV or V, between 20,000 and 25,000 cm^{-1}, which find no place in either part of the present analysis. These lines can just be brought out in the high-temperature furnace, which indicates that the lower level involved is considerably higher than 25,000 cm^{-1}. This would make the upper levels above the ionization limit.

These levels may be attributed to other terms of the configuration f^7s^2, of which the lowest should be ${}^6G^o$, ${}^6H^o$, and ${}^6I^o$ (or some of them). The triads with which these combine involve terms of types ${}^6FGHIK$. These will be subject to auto-ionization only if a continuum of similar terms of even parity exists beyond the limit a^7S^o of *Eu* II, which cannot occur for ${}^6G$ or ${}^6I$. Above the a^7D^o level of *Eu* II continua of all types occur, but this limit is 17,000 cm^{-1} above a^9S^o and 62,000

above a^8S^o of *Eu* I. Many sharp even levels may exist below this limit.

No significant differences have been found among the lines of this group. The data are probably insufficient.

4. SERIES AND IONIZATION POTENTIAL

There are six pairs of terms (Table 1) which are in series, with a running s-, d-, or f-electron. Rydberg's formula applied to these gives limits ranging from 45,700 to 46,300 for a^9S^o of *Eu* II, and

TABLE 2

VALUES OF Δn^*

Term	*Sr* I	*Cd* I	*Ba* I	*Hg* I	*Ra* I	Term	*Eu* I	
							45700	45800
^1S						a^8S^o		
^3S	1.000	1.053	1.014	0.999	1.029	$e^{10}S^o$	0.996	0.990
^3S	1.043	1.034	1.045	1.033	1.046	$f^{10}S^o$	*1.046*	*1.033*
^3P						$z^{10}P$		
^3P	1.149	1.173	1.196	1.183	1.196	$x^{10}P$	1.259	1.248
^3D						$a^{10}D^o$		
^3D	1.182	....	1.350	....	1.261	$e^{10}D^o$	1.288	1.277
^3D	1.023	1.009	1.005	1.009	....	$f^{10}D^o$	*1.026*	*1.009*
^3F						$y^{10}F^o$		
^3F	0.992	0.997	1.031	0.998	0.980	$x^{10}F^o$	*0.997*	*0.970*

47,800 for a^7S, which is higher by 1669. To obtain a better value of the ionization potential we may attempt to make the values Δn^* of the difference of the Rydberg denominators for successive terms the same as for other two-electron spectra in which the limits are fixed by long series. Table 2 gives the values of the latter (for the middle components of multiple terms) and also those obtained for two assumed values of $a^8S^o - a^9S^o$ in europium.

The three pairs of values of Δn^* given in italics are the most sensitive to a change in the limit. To make these identical with the means for the other five spectra, the limit should be at 45,750,

45,780, and 45,690. The mean 45,740 may be adopted, corresponding to an ionization potential of 5.64 volts, with a probable error less

TABLE 3

VALUES OF n^*

Electron	Limit	Term	n^*	Limit	Term	n^*
6s	a^9S^o	a^8S^o	1.549	a^7S^o	a^8S^o	1.521
7s	a^9S^o	$e^{10}S^o$	2.542	a^7S^o	f^8S^o	2.603
	a^9S^o	e^8S^o	2.600	a^7S^o	e^6S^o	2.557
8s	a^9S^o	$f^{10}S^o$	3.584	a^7S^o	i^8S^o	3.666
	a^9S^o	g^8S^o	3.476	a^7S^o	f^6S^o	3.598
6s	a^9D^o	$a^{10}D^o$	1.601	a^7D^o	b^8D^o	1.594
	a^9D^o	a^8D^o	1.643	a^7D^o	a^6D^o	1.589
6p	a^9S^o	$z^{10}P$	1.876	a^7S^o	y^8P	2.062
	a^9S^o	z^8P	1.919	a^7S^o	z^6P	1.922
7p	a^9S^o	$x^{10}P$	3.131	a^7S^o	u^8P	3.186
	a^9S^o	w^8P	3.133	a^7S^o	x^6P	(3.651)
6p	a^9D^o	$z^{10}F$	2.021	a^7D^o	y^8F	2.146
	a^9D^o	$z^{10}D$	2.109	a^7D^o	y^8D	2.047
	a^9D^o	$y^{10}P$	2.163	a^7D^o	v^8P	2.032
	a^9D^o	z^8F	2.137	a^7D^o	z^6F	2.031
	a^9D^o	z^8D	2.086	a^7D^o	z^6D	2.120
	a^9D^o	x^8P	2.236	a^7D^o	y^6P	2.145
5d	a^9S^o	$a^{10}D^o$	1.837	a^7S^o	b^8D^o	1.991
	a^9S^o	a^8D^o	1.906	a^7S^o	a^6D^o	1.982
6d	a^9S^o	$e^{10}D^o$	3.120	a^7S^o	g^8D^o	3.115
	a^9S^o	e^8D^o	3.202	a^7S^o	e^6D^o	3.187
7d	a^9S^o	$f^{10}D^o$	4.130	a^7S^o	f^6D^o	4.204
	a^9S^o	h^8D^o	4.192			
5f	a^9S^o	$y^{10}F$	3.918	a^7S^o	x^8F	3.912
				a^7S^o	y^6F	3.903
6f	a^9S^o	$x^{10}F$	4.900	....	....	....

than 0.01 volt. This corresponds to $a^9S^o_4$ of *Eu* II. The higher limits are $a^7S^o_3$ 47,409, $a^9D^o_4$ 56,053, and $a^7D^o_3$ 62,881.

With these limits we find for the principal terms of *Eu* I the values of n^* given in Table 3. These values confirm the assignment of the terms to the series position except for x^6P, which should be at about

36,000. There are some levels near this value which combine strongly with a^6D^o, but nothing like a multiplet.

The discordances for g^8S^o, i^8S^o can be explained by repulsion by h^8S^o.

The configuration 5d7p should give ^{10}PDF, ^8PDF near 45,000, close to ionization; 5d7s gives levels near 41,000. Some of the unclassified even levels may come from these or from $(6p)^2$, while those near 34,000 may be parts of the missing F^o and G^o terms from $(5d)^2$.

5. MULTIPLETS

The multiplets in this spectrum are remarkably regular for an element of high atomic number, and a summary exhibit of the line intensities may be appropriate, especially as these have all been estimated on a homogeneous scale. Table 4 gives the data for the multiplets arising from the low odd terms, and Table 5 for the low even terms. A dash (–) denotes that the line is absent, "m" that it is masked, and "i" that it is in the infrared. For a few lines absent from, or masked in the arc, the furnace intensities are given in square brackets. The combinations of the 6s5d and 6s6p configurations are in the remote infrared, some of them beyond 20 μ.

For the high multiplicities the intensities approach those in Zeeman patterns, as they should theoretically do. The weakness of one diagonal component in the PP^o and DD^o groups is striking. There is a curious tendency for the middle diagonal to be faint in many of the intersystem groups.

Many lines in this spectrum are conspicuously widened. The relation of the observed degree of widening (denoted by the letters "w_1" to "w_4" in Table 2 of the previous communication[4]) to the place in the present analysis has been investigated. Practically all the lines involving terms belonging to the electron configurations 6s7d, $(6p)^2$, 6s5f, and 6s6f are considerably or greatly widened, and so are many lines involving 6s7p. This widening is evidently of the familiar type associated with large total quantum number. The strong lines of the principal multiplets are rarely recorded as widened. Widening due to hyperfine structure is probably present, but the existing data do not suffice to disentangle it.

TABLE 4

Multiplets of *Eu* I—Low Odd Terms

	$6s^2$ a^8S^o	6s5d $a^{10}D^o$					6s5d a^8D^o					6s5d b^8D^o					6s5d a^6D^o				
	3½	6½	5½	4½	3½	2½	5½	4½	3½	2½	1½	5½	4½	3½	2½	1½	4½	3½	2½	1½	½
5d6p $z^{10}F$ 7½		5000																			
6½		1200	2500				150					–									
5½		100	1500	1200			20	200				i	i				i				
4½	–		300	1500	600		8	4	200			i	i	i			i	i			
3½	40*			400	800	150		8	–	100			i	i	i		i	i	i		
2½	2				1000	1000			40	5	150			i	i	i		i	i	i	
1½						1200				30	50				i	i			i	i	i
5d6p $z^{10}D$ 6½		1500	1200				125					5									
5½		1000	600	1200			15	40				–	–				–				
4½	70*		1500	3	1200		6	4	20			–	2	–			–	–			
3½	300			800	125	800		10	1	15			–	4	3		–	–	–		
2½	5				800	1200			60	m	40			8	–	4		1	–	–	
5d6p $y^{10}P$ 5½		2000	1200	400			300	1				15	8				m				
4½	40		1000	2000	800		8*	60	25*			1	4	4			–	–			
3½	6			300	1200	1500		6	1	m			3	–	–		–	–	–		
6s7p $x^{10}P$ 5½		600†	300	50			3	1				–	–				–				
4½	300		200	200	50		15	1	1			3	–	–			–	–			
3½	10			250	150	100		–	–	–			–	–	–		–	–	–		

* Blend with *Eu* II.
†Blend.

TABLE 4—*Continued*

		$6s^2$ a^8S^o	$6s5d$ $a^{10}D^o$					$6s5d$ a^8D^o					$6s5d$ b^8D^o					$6s5d$ a^6D^o				
		3½	6½	5½	4½	3½	2½	5½	4½	3½	2½	1½	5½	4½	3½	2½	1½	4½	3½	2½	1½	½
6s5f $y^{10}F$	7½		150																			
	6½		m	50				—					m									
	5½		5†	10	m			—	—				—	—				—				
	4½	—		4*	15	25		—	—	—			—	—	—			m	1			
	3½	—			[6]‡	60	10*		—	—	—			—	—	—		—	—	—		
	2½	—				20	30			—	—	—			—	—	—		—	—	—	
	1½						30				—	—				—	—			—	—	—
6s6f $x^{10}F$	7½		8*																			
	6½		2	5				—					6									
	5½		5	10	15			—	—				—	—				—				
	4½	1†		8	10	8		—	—	—			—	—	—			4	—			
	3½	1†			8	6	—		—	—	—			—	—	—		—	—	—		
	2½	—				5*	2			—	—	—			—	—	—		—	—	—	
	1½						4				—	—				—	—			—	—	—
5d6p z^8F	6½		1200	50†				1500					20									
	5½		—	500	20			400	800				1	40				—				
	4½	100		8	m	1		50	600	400			8	1	50			—	—			
	3½	—			6	8	2		150	600	m			3	5	30		—	—	—		
	2½	—				m	m			300	400	m			—	15	10		—	—	—	
	1½						1				250	300				2?	10			—	—	—
	½											250					20				—	—
5d6p z^8D	5½		800	1	150			1500	600				200	10				8				
	4½	30		500	10	60		600	600	600			30	40	12			15	4			
	3½	50			600	m	m		400	100	600			20	4	3		6	2	4		
	2½	20				600	5			300	2	500			5	—	3		10	—	2	
	1½						40				300	300				2	10			4	2	—

‡ Brackets denote furnace intensity.

TABLE 4—*Continued*

	$6s^2$ a^8S^o	$6s5d$ $a^{10}D^o$					$6s5d$ a^8D^o					$6s5d$ b^8D^o					$6s5d$ a^6D^o				
	3½	6½	5½	4½	3½	2½	5½	4½	3½	2½	1½	5½	4½	3½	2½	1½	4½	3½	2½	1½	½
5d6p																					
x^8P 4½	50		3	8	1		500	150	50†			2	1†	–			1	–			
3½	6*			4	1	8		400	300	80			10	2	–		2	–	–		
2½	30				4	10†			250	250	250			–	4	–		–	–	–	
6s7p																					
w^8P 4½	250		50	30	4	.	20	5?	2			20	4	–			3*	–			
3½	200			4	1	1		[50?]	20	6			4	4	2		2	2	–		
2½	150				1	–			3	25	8			3	8	8		1	1†	–	
5d6p																					
y^8F 6½		–	–				25					300									
5½		–	–	–			–	10				12	150				15				
4½	–		–	–	–		–	–	12*			–	30	40			–	15			
3½	–			–	–	–		–	–	1			2	40	12		4	1	6		
2½	–				–	–			–	–	–			–	12	1		2	m	4	
1½						–				–	5				3	20			m	–	2
½											[1]					20				1	1
5d6p																					
y^8D 5½		200	12*	3*			200	40				400	m				1				
4½	40		4*	–	–		100	300	60			6	50	50			10	–			
3½	50			30	4*	–		m	150	100			20	15	125		12	3	–		
2½	18				6	–			15	3	10			40	–	60		20	–	m	
1½						4				40	15				150†	60			15	–	m
5d6p																					
v^8P 4½	40		200	10	–		150	1	3			50	10	15			–	1			
3½	60			1	20	12		200	10	4			40	–	8		12	1	8		
2½	40				8	1			40	20	5			8	–	4		m	–	1	

TABLE 4—*Continued*

		6s² a⁸S°	6s5d a¹⁰D°					6s5d a⁸D°					6s5d b⁸D°					6s5d a⁶D°				
		3½	6½	5½	4½	3½	2½	5½	4½	3½	2½	1½	5½	4½	3½	2½	1½	4½	3½	2½	1½	½
6s7p u⁸P	4½	15		200	25	6		150	15	8			80	15	m			–	1			
	3½	20			30*	50	25		100	6	1			10	–	m		3	m	–		
	2½	80				3	–			40	40	40			1	1	–		1	–	–	
6s5f x⁸F	6½		–	–				50					10									
	5½		–	–	–			8†	60				–	3				–				
	4½	–		–	–	–		m	30	50			–	2	m			–	–			
	3½	–			–	–	–		m	40	20			–	1	1†		–	–	–		
	2½	–				–	–			m	30	6			–	1†	–		2	1	–	
	1½						–				8	10				–	1			1	–	–
	½											8					–				1	–
5d6p z⁶F	5½		–	–	–			m	10				10	30				150				
	4½	2		–	–	–		4	–	8			6	10	8			10	150			
	3½	2			–	–	–		3	m	15			m	15	3		–	30	150		
	2½	–				m	–			10†	2	6			10	6	1		–	150†	60	
	1½						–				6	4				m	m			–	40	20
	½											m					m				–	10
5d6p z⁶D	4½	10		–	–	–		–	–	–			–	2	–			200	20			
	3½	3			–	–	–		m	–	–			–	1	–		m	150	50		
	2½	–				–	–			–	–	–			1	–	m		30	50	60	
	1½						–				m	–				–	1			40	2	25
	½											–					1				30	5

TABLE 4—*Continued*

	$6s^2$ a^8S^0	$6s5d$ $a^{10}D^0$					$6s5d$ a^8D^0					$6s5d$ b^8D^0					$6s5d$ a^6D^0				
	$3\frac{1}{2}$	$6\frac{1}{2}$	$5\frac{1}{2}$	$4\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$	$5\frac{1}{2}$	$4\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$	$1\frac{1}{2}$	$5\frac{1}{2}$	$4\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$	$1\frac{1}{2}$	$4\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$	$1\frac{1}{2}$	$\frac{1}{2}$
$5d6p$																					
y^6P $3\frac{1}{2}$	–			–	–	–		–	–	–			4	–	4		250	6	3		
$2\frac{1}{2}$	–				–	–			5	–	–			2	–	4		30	30	10	
$1\frac{1}{2}$						–				–	4				m	1			30	40	5
x^6P $3\frac{1}{2}$	–			–	–	–		5*	–	–			25	10	1		20	2	1		
$2\frac{1}{2}$	–				–	–			–	15*	–			8	–	1		10	8	3	
$1\frac{1}{2}$						–				3	2				1	3			1	3	5
$6s5f$																					
y^6F $5\frac{1}{2}$		–	–	–			–	–				–	12				200				
$4\frac{1}{2}$	–		–	–	–		–	–	1			–	1	4			·6	60			
$3\frac{1}{2}$	–			–	–	–		–	–	–			–	–	–		–	50?	50		
$2\frac{1}{2}$	–				–	–			–	–	–			–	1	–		1	15	20	
$1\frac{1}{2}$						–				–	–				–	1			m	m	5
$\frac{1}{2}$											m					–				3*	6
$100_{3\frac{1}{2},\,4\frac{1}{2}}$	60			–	–			1	–				i	i			i	i			
$101_{3\frac{1}{2}}$	30			200	400	80		6	–	80			i	i	i		i	i	i		
$102_{4\frac{1}{2}}$	–		1	3†	1		–	–	–			i	i	i			i	i			
$103_{2\frac{1}{2},\,3\frac{1}{2}}$	20				1	1			–	–				i	i			i	i		
$104_{3\frac{1}{2},\,4\frac{1}{2}}$	20			1	1†			–	–				i	i				i			
$105_{4\frac{1}{2}}$	50		2	m	m		–	–	–			i	i	i			i	i			
$106_{3\frac{1}{2},\,4\frac{1}{2}}$	600			1†	–			–	–				i	i			–	–			
107	80				–				–					–				–			
$108_{3\frac{1}{2},\,4\frac{1}{2}}$	10			–	m			1	m				–	–			–	–			
$109_{3\frac{1}{2}}$	40 ·			1	–	–		6	3	8			4	–	–		–	–	–		
$110_{4\frac{1}{2},\,5\frac{1}{2}}$			–	m			20?	5				6	–				–				
$111_{5\frac{1}{2}}$		40	1	100			600	200				25	5				2				

TABLE 4—*Continued*

	$6s^2$ a^8S^0	$6s5d$ $a^{10}D^0$					$6s5d$ a^8D^0					$6s5d$ b^8D^0					$6s5d$ a^6D^0				
	3½	6½	5½	4½	3½	2½	5½	4½	3½	2½	1½	5½	4½	3½	2½	1½	4½	3½	2½	1½	½
$112_{2\frac{1}{2}}$	200				4	15			10	[5]	4			3	–	–		–	–	–	
$113_{3\frac{1}{2}}$	1000			600	80	250		20	[5?]	20			6	–	–		1	–	–		
$114_{4\frac{1}{2}}$	30		1	–	–		–	2	2			–	–	–			–	–			
$115_{4\frac{1}{2}}$	500		1	150	150†		–	100	80			30	3	6			–	–			
$116_{4\frac{1}{2}, 5\frac{1}{2}}$			40	1			40	60				–	3				–				
$117_{5\frac{1}{2}, 6\frac{1}{2}}$		5	–				5					–									
$118_{3\frac{1}{2}}$	100			4	3	5		1*	–	–			–	1	–		–	–	–		
$119_{3\frac{1}{2}}$	20			2	12	m		–	3	–			–	–	–		m	–	2		
$120_{3\frac{1}{2}}$	4			6	m	125		8	–	m			–	–	–		1	–	–		
$121_{2\frac{1}{2}}$	–				4	–			–	–	3			–	–	–		m	–	–	
$122_{4\frac{1}{2}}$	15		8	3	1		[50?]	6	1			–	–	–			–	–			
$123_{3\frac{1}{2}}$	4			–	–	–		5	1	–			1	1	2		–	2	2		
$124_{2\frac{1}{2}}$	5				–	–			1	1	–			–	–	1		–	15	2	
$125_{1\frac{1}{2}}$						2				1	–				–	–			1	4	5
$126_{1\frac{1}{2}}$						–				1	–				–	–			2	4	2
$127_{1\frac{1}{2}, 2\frac{1}{2}}$						30				–	m				2	5			–	–	
$128_{2\frac{1}{2}}$	10				100	–			–	4	–			6	–	1		1	–	–	
$129_{3\frac{1}{2}, 4\frac{1}{2}}$	3			–	–			1	1				6	3			2	60			
$130_{2\frac{1}{2}, 3\frac{1}{2}}$	–				–	–			–	–				–	–			6	15		
$131_{1\frac{1}{2}}$						–				–	–				1	–			–	80*	5
$132_{5\frac{1}{2}}$		300	6	–			150	10				80	12				–				
$133_{2\frac{1}{2}}$	2				40	6			–	–	–			–	–	1		–	1	1	
$134_{4\frac{1}{2}}$	5		–	80	10		2	–	–			–	1	1			m	–			
$135_{2\frac{1}{2}}$	8				8	[40]			–	4	20			–	–	1		–	–	–	
$136_{5\frac{1}{2}, 6\frac{1}{2}}$		8	25				–					–									
$137_{2\frac{1}{2}, 3\frac{1}{2}}$	–				–	3			–	4				1	–			4	20		
$138_{3\frac{1}{2}}$	–			15	8	–		6	3	–			–	m	2		1	20	–		
$139_{2\frac{1}{2}, 3\frac{1}{2}}$	–				–	–			–	–				–	–			3	3		
$140_{3\frac{1}{2}}$	–			25	8	10		–	–	–			–	m	–		1	15	–		
$141_{4\frac{1}{2}}$	15		–	–	–		15	10	–			–	4	4			40	–			

TABLE 4—*Continued*

	$6s^2$ a^8S^o	$6s5d$ $a^{10}D^o$					$6s5d$ a^8D^o					$6s5d$ b^8D^o					$6s5d$ a^6D^o				
	3½	6½	5½	4½	3½	2½	5½	4½	3½	2½	1½	5½	4½	3½	2½	1½	4½	3½	2½	1½	½
$142_{5½}$		6	[2]	–			15	–				8	2				3†				
$143_{3½}$	6			20	1	125		–	–	–			–	–	–		–	1	–		
$144_{3½}$				10	5†	40		4	5	3			–	–	–		–	–	–		
$145_{4½}$	3		–	–	–		3	4*	–			–	–	–			8	3			
$146_{3½}$	2			6	4	4		40	12	4			m	–	–		–	3	2		
$147_{4½}$	–		10	8†	60*		–	5	–			–	–	–			–	–			
$148_{3½}$?	–			30	2	–		–	–	–			m	–	3†		1	–	–		
$149_{5½, 6½}$		40	8				–					–									
$150_{3½}$	1			–	–	–		–	–	–			–	m	–		2	20	5		
$151_{4½}$	1		25	15	50		–	–	–			–	–	m			–	–			
$152_{5½, 6½}$		10	40				–					–									
$153_{4½}$	12		5	15	20		–	m	–			–	2	–			m	–			
$154_{3½}$	–			10	10	5?		5	–	–			m	–	–		1	–	–		
$155_{3½}$	3			–	8	–		–	20	8			–	–	2		1	m	–		
$156_{5½}$		6	10	40			–	3				–	–				–				
$157_{4½}$	4		–	–	–		6	–	–			20	1?	10			–	–			
$158_{5½, 6½}$		40	80				–					–									
$159_{2½}$					2	3			4	–	1			–	–	–		–	–	–	
$160_{4½, 5½}$			–	–			60	15				–	–				2				
$161_{4½, 5½}$			4	5			–	–				–	–				–				
$162_{5½, 6½}$		6	10?				–					–									

TABLE 5

MULTIPLETS OF *Eu* I—LOW EVEN TERMS

		6s6p $z^{10}P$			6s6p z^8P			6s6p y^8P			6s6p z^6P		
		5½	4½	3½	4½	3½	2½	4½	3½	2½	3½	2½	1½
6s7s $e^{10}S^o$	4½	1000	2500	2500	250	50		i	i		6		
$(5d)^2$ $e^{10}P^o$	5½	800	600		m			4					
	4½	300	8	600	–	5		i	–		–		
	3½		1200	600	20	30	15	i	i	i	3	–	
6s6d $e^{10}D^o$	6½	3000											
	5½	1000	1500		m			i					
	4½	200	2000	600	15	3		i	i		–		
	3½		600?	1500	8	8	–	i	i	i	1	–	
	2½			1800		8	–		i	i	3	–	–
6s8s $f^{10}S^o$	4½	50	80	100	3	3		–	–		–		
6s7d $f^{10}D^o$	6½	50											
	5½	15	50		–			–					
	4½	8	50	20	–	–		–	–		–		
	3½		20	25	–	m	–	–	m	–	–	–	
	2½			25		3	–		–	–	–	–	–
$(6p)^2$ $f^{10}P^o$	5½	150	100		25			1					
	4½	80	–	50	2	12		m	m		–		
	3½		100	20	15	6	5	–	1	m	–	–	
$(6s)^2$ a^8S^o	3½		3000	1500	2500	150	300	10000	8000	7000	2500	1200	
6s7s e^8S^o	3½		400	80	500	600	800	i	i	i	500	40	
6s7s f^8S^o	3½		50	50	200	60	80	i	i	i	15?	4	
$(5d)^2$ e^8P^o	4½	–	40	–	300	1		12	40		4		
	3½		6	m	200	50	200	8	–	20	–	–	
	2½			40		400	2		10	40	60?	–	–
6s6d e^8D^o	5½	–	30		300			200					
	4½	–	4	–	125	10		30	40		10		
	3½		–	3	100?	150†	2	4	15	25	8	1	
	2½			–		1	1		2	5	1	1	–
	1½						1			–		–	–

† Blend.

TABLE 5—*Continued*

		6s6p $z^{10}P$			6s6p z^8P			6s6p y^8P			6s6p z^6P		
		5½	4½	3½	4½	3½	2½	4½	3½	2½	3½	2½	1½
$(5d)^2$ f^8D^o	5½	–	40		300			20					
	4½	–	20	40	60	500		m	3		100		
	3½		–	3	1	8*	200	1	80	15	100	1	
	2½			3		40	300		20	40	–	30	12
	1½						125			100		20	–
6s6d g^8D^o	5½	1	50		600			1000					
	4½	m	30	20	300	125		150	200		2*		
	3½		15	20	3	500	200	8	150	20	1	2	
	2½			10		150	300		50	100	2	m	–
	1½						400			100		m	–
6s7d h^8D^o	5½	–	–		80			5					
	4½	–	m	–	40*	25*		2	2		m		
	3½		–	–	8*	12	6	–	1	2	m	–	
	2½			–		–	10		–	2	–	–	–
6s8s g^8S^o	3½		12	–	60	4	8	30	3	20	30	15	
$(5d)^2$ h^8S^o	3½		8	6	20	200	[25]‡	10	12	8	150	80	
6s8s i^8S^o	3½		–	–	5	[4]	5	8	6	4	15	4	
6s7s e^6S^o	2½			1		800	300		i	i	400	300	500
$(5d)^2$ e^6P^o	3½		4	–	4	15	2	3†	–	2	300	100?	
	2½			–		60	15		m	2	250	5	300
	1½						50			2		600†	[15]
6s6d e^6D^o	4½	m	–	–	2	300		–	15		800		
	3½		–	–	–	25	20	–	10	3	150	[300]	
	2½			–		–	10		1	3	10	300	60
	1½						2			2		40	100
	½												80
6s7d f^6D^o	4½	–	–	–	–	15		–	–		20		
	3½		–	–	–	2	4	–	–	–	12	m	
	2½			–		–	15		–	–	–	12	m
	1½						–			–		[4]	25

* Blend with *Eu* II.

‡ Brackets denote furnace intensity.

TABLE 5—*Continued*

	6s6p $z^{10}P$			6s6p $z^{8}P$			6s6p $y^{8}P$			6s6p $z^{6}P$		
	$5\frac{1}{2}$	$4\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$	$4\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$	$1\frac{1}{2}$
6s8s f^6S^o $2\frac{1}{2}$			–		12	5		–	–	15	20	10
$1^{\circ}_{2\frac{1}{2}}$			–		–	20		–	–	2	–	2
$2^{\circ}_{1\frac{1}{2}, 2\frac{1}{2}}$						10			–		1	1
$3^{\circ}_{3\frac{1}{2}, 4\frac{1}{2}}$		–	12	4	100		–	–		8		
4°		200		8			4					
$5^{\circ}_{3\frac{1}{2}}$		m	–	4	20	4	–	–	–	–	–	
$6^{\circ}_{2\frac{1}{2}, 3\frac{1}{2}}$		–	–	–	8	15*	–	–	–	–	–	–
$7^{\circ}_{3\frac{1}{2}}$		–	10	8	6	10	1	m	–	–	–	
$8^{\circ}_{4\frac{1}{2}, 5\frac{1}{2}}$	10	–	–	10	–		2	–		–		
$9^{\circ}_{4\frac{1}{2}}$	15	–	30	8†	10		–	m		–		
$10^{\circ}_{3\frac{1}{2}, 4\frac{1}{2}}$		–	–	4	8†		2	30		–		
$11^{\circ}_{3\frac{1}{2}, 4\frac{1}{2}}$		15	3	–	–		–	–		–		
$12^{\circ}_{4\frac{1}{2}, 5\frac{1}{2}}$	30	8		20			–					
$13^{\circ}_{4\frac{1}{2}, 5\frac{1}{2}}$	30	5		10			–					
$14^{\circ}_{3\frac{1}{2}, 4\frac{1}{2}}$		–	–	8	6		–	–		2		
$15^{\circ}_{4\frac{1}{2}, 5\frac{1}{2}}$	[4]	–		30*			m					
$16^{\circ}_{3\frac{1}{2}, 4\frac{1}{2}}$		20	–	150	30		–	–		–		
$17^{\circ}_{4\frac{1}{2}, 5\frac{1}{2}}$	25	–		30			–					
18°		–		100			8					
$19^{\circ}_{3\frac{1}{2}, 4\frac{1}{2}}$		4	–	–	5*		–	–		–		

6. TERM TABLE

The terms and undesignated energy-levels which result from the present analysis are given in Table 6. They are arranged in order of increasing energy, but the components of a clearly established multiple term have been kept together for convenience. Such terms are listed in the order of their lowest levels. Odd terms are printed in italics. The unclassified odd levels are denoted by numbers from 1° to 19°; the even ones, from 100 to 162. A few levels, whose reality is not quite assured, are marked with colons. All the sextet terms are inverted, while many of the octet terms going to the limit a^7S^o have both positive and negative intervals.

7. LIST OF CLASSIFIED LINES

All the lines which have been classified are found in Table 7, which gives in successive columns (1) the observed wave length, (2) the difference $\Delta\lambda$ (O – C) between this and the value computed from the adopted terms, (3) the arc intensity, (4) the temperature class,

TABLE 6

TERMS OF *Eu* I

Desig.	Level	Diff.
$a^8S^\circ_{3\frac12}$	*0.00*	
$a^{10}D^\circ_{2\frac12}$	*12923.72*	
$3\frac12$	*13048.90*	125.18
$4\frac12$	*13222.04*	173.14
$5\frac12$	*13457.21*	235.17
$6\frac12$	*13778.68*	321.47
$z^{10}P_{3\frac12}$	14067.86	
$4\frac12$	14563.57	495.71
$5\frac12$	15581.58	1018.01
$a^8D^\circ_{1\frac12}$	*15137.72*	
$2\frac12$	*15248.76*	111.04
$3\frac12$	*15421.25*	172.49
$4\frac12$	*15680.28*	259.03
$5\frac12$	*16079.76*	399.48
$z^8P_{2\frac12}$	15890.53	
$3\frac12$	15952.31	61.78
$4\frac12$	16611.79	659.48
$z^6P_{3\frac12}$	17340.65	
$2\frac12$	17707.42	− 366.77
$1\frac12$	17945.49	− 238.07
$a^6D^\circ_{4\frac12}$	*19273.24*	
$3\frac12$	*19364.50*	− 91.26
$2\frac12$	*19462.05*	− 97.55
$1\frac12$	*19543.69*	− 81.64
$\frac12$	*19599.16*	− 55.47
$b^8D^\circ_{5\frac12}$	*19447.19*	
$4\frac12$	*19631.26*	− 184.07
$3\frac12$	*19712.15*	− 80.89
$2\frac12$	*19763.78*	− 51.63
$1\frac12$	*19794.21*	− 30.43
$y^8P_{2\frac12}$	21444.58	
$3\frac12$	21605.17	160.59
$4\frac12$	21761.26	156.09
$100_{3\frac12,\,4\frac12}$	27852.90	
$z^{10}F_{1\frac12}$	28519.97	
$2\frac12$	28667.33	147.36
$3\frac12$	28918.12	250.79
$4\frac12$	29186.32	268.20
$5\frac12$	29612.69	426.37
$6\frac12$	30211.09	598.40
$7\frac12$	30923.71	712.62
$e^{10}S^\circ_{4\frac12}$	*28763.82*	

Desig.	Level	Diff.
$101_{3\frac12}$	28827.83	
$102_{4\frac12}$	29045.75	
$103_{2\frac12,\,3\frac12}$	29124.78	
$e^8S^\circ_{3\frac12}$	*29517.86*	
$104_{3\frac12,\,4\frac12}$	29809.23	
$105_{4\frac12}$	29838.59	
$106_{3\frac12,\,4\frac12}$	29982.44	
107	30091.34	
$e^6S^\circ_{2\frac12}$	*30619.49*	
$108_{3\frac12,\,4\frac12}$	30642.54	
$109_{3\frac12}$	30783.64	
$z^8D_{1\frac12}$	30800.71	
$2\frac12$	30797.96	− 2.75
$3\frac12$	30841.89	43.93
$4\frac12$	30901.74	59.85
$5\frac12$	31014.48	112.74
$110_{4\frac12,\,5\frac12}$	30819.14	
$111_{5\frac12}$	30819.44	
$z^{10}D_{2\frac12}$	30945.07	
$3\frac12$	31138.11	193.04
$4\frac12$	31382.56	244.45
$5\frac12$	31725.93	343.37
$6\frac12$	32117.10	391.17
$112_{2\frac12}$	31107.28	
$113_{3\frac12}$	31116.38	
$f^8S^\circ_{3\frac12}$	*31217.30*	
$114_{4\frac12}$	31553.76	
$z^8F_{\frac12}$	31735.76	
$1\frac12$	31787.71	51.95
$2\frac12$	31876.04	88.33
$3\frac12$	32003.17	127.13
$4\frac12$	32184.59	181.42
$5\frac12$	32418.38	233.79
$6\frac12$	32761.74	343.36

TABLE 6—*Continued*

Desig.	Level	Diff.
$e^{10}P^{\circ}_{3\frac{1}{2}}$	*31848.81*	
4½	*32209.88*	361.07
5½	*32480.95*	271.07
$115_{4\frac{1}{2}}$	32130.25	
$116_{4\frac{1}{2},\,5\frac{1}{2}}$	32326.73	
$y^{10}P_{3\frac{1}{2}}$	32398.18	
4½	32596.23	198.05
5½	32948.41	352.18
$117_{5\frac{1}{2},\,6\frac{1}{2}}$	32598.00	
$118_{3\frac{1}{2}}$	32681.13	
$x^{8}P_{2\frac{1}{2}}$	33786.47	
3½	34101.78	315.31
4½	34725.71	623.93
$119_{3\frac{1}{2}}$	33879.10	
$120_{3\frac{1}{2}}$	33908.77	
$121_{2\frac{1}{2}}$	33964.87:	
$1^{\circ}_{2\frac{1}{2}}$	*34065.73*	
$2^{\circ}_{1\frac{1}{2},\,2\frac{1}{2}}$	*34081.16*	
$3^{\circ}_{3\frac{1}{2},\,4\frac{1}{2}}$	*34126.42*	
4°	*34306.36*	
$x^{10}P_{5\frac{1}{2}}$	34316.97	
4½	34546.06	− 229.09
3½	36005.68	−1459.62
$w^{8}P_{4\frac{1}{2}}$	34366.13	
3½	34561.73	− 195.60
2½	34555.82	5.91
$e^{10}D^{\circ}_{2\frac{1}{2}}$	*34422.94*	
3½	*34440.50*	17.56
4½	*34466.80*	26.30
5½	*34505.55*	38.75
6½	*34544.59*	39.04
$122_{4\frac{1}{2}}$	34738.80	
$e^{8}D^{\circ}_{1\frac{1}{2}}$	*34913.59:*	
2½	*34967.69*	54.10
3½	*35033.98*	66.29
4½	*35106.94*	72.96
5½	*35204.58*	97.64

Desig.	Level	Diff.
$e^{8}P^{\circ}_{2\frac{1}{2}}$	*35174.20*	
3½	*35053.23*	− 120.97
4½	*35280.43*	227.20
$f^{8}D^{\circ}_{1\frac{1}{2}}$	*35282.22*	
2½	*35398.16*	115.94
3½	*35377.94*	− 20.22
4½	*35813.47*	435.53
5½	*35460.74*	− 352.73
$z^{6}F_{5\frac{1}{2}}$	35453.23	
4½	35731.56	− 278.33
3½	36080.90	− 349.34
2½	36284.75	− 203.85
1½	36501.65	− 216.90
½	36586.37	− 84.72
$123_{3\frac{1}{2}}$	35612.37	
$124_{2\frac{1}{2}}$	35703.54	
$125_{1\frac{1}{2}}$	35762.12	
$126_{1\frac{1}{2}}$	35799.09	
$127_{1\frac{1}{2},\,2\frac{1}{2}}$	35941.51	
$g^{8}D^{\circ}_{1\frac{1}{2}}$	*36045.39*	
2½	*36072.62*	27.23
3½	*36097.64*	25.02
4½	*36219.00*	121.36
5½	*36242.34*	23.34
$128_{2\frac{1}{2}}$	36052.59	
$129_{3\frac{1}{2},\,4\frac{1}{2}}$	36071.71	
$130_{2\frac{1}{2},\,3\frac{1}{2}}$	36107.85:	
$131_{1\frac{1}{2}}$	36334.52	
$v^{8}P_{2\frac{1}{2}}$	36410.91	
3½	36381.44	− 29.47
4½	36548.83	167.39
$u^{8}P_{2\frac{1}{2}}$	36441.77	
3½	36600.83	159.06
4½	36504.49	− 96.34
$e^{6}D^{\circ}_{4\frac{1}{2}}$	*36566.64*	
3½	*36589.28*	− 22.64
2½	*36608.42*	− 19.14
1½	*36622.00*	− 13.58
½	*36630.46*	− 8.46

TABLE 6—*Continued*

Desig.	Level	Diff.
$\mathrm{y^8D_{1½}}$	36586.35	− 2.10
$_{2½}$	36584.25	116.08
$_{3½}$	36700.33	189.23
$_{4½}$	36889.56	− 22.52
$_{5½}$	36867.04	
$\mathrm{g^8S^{\circ}_{3½}}$	*36659.31*	
$132_{5½}$	37034.83	
$133_{2½}$	37093.74	
$134_{4½}$	37126.08	
$\mathrm{f^{10}S^{\circ}_{4½}}$	*37195.76*	
$135_{2½}$	37266.29	
$136_{5½,\,6½}$	37392.64	
$137_{2½,\,3½}$	37502.42	
$138_{3½}$	37504.67	
$139_{2½,\,3½}$	37574.26:	
$\mathrm{e^6P^{\circ}_{3½}}$	*37584.98*	− 452.36
$_{2½}$	*38037.34*	− 208.32
$_{1½}$	*38245.66*	
$140_{3½}$	37589.35	
$141_{4½}$	37591.12	
$142_{5½}$	37800.42	
$143_{3½}$	37812.80	
$144_{3½}$	37851.66	
$145_{4½}$	37908.73	
$\mathrm{h^8S^{\circ}_{3½}}$	*37993.98*	
$146_{3½}$	38072.62	
$\mathrm{z^6D_{4½}}$	38167.10	− 244.65
$_{3½}$	38411.75	− 45.51
$_{2½}$	38457.26	− 73.66
$_{1½}$	38530.92	− 33.06
$_{½}$	38563.98	
$147_{4½}$	38262.81	

Desig.	Level	Diff.
$148_{3½}$?	38290.93	
$149_{5½,\,6½}$	38292.17	
$150_{3½}$	38305.20	
$151_{4½}$	38360.67	
$152_{5½,\,6½}$	38534.17	
$\mathrm{y^{10}F_{1½}}$	38565.73	6.90
$_{2½}$	38572.63	− 3.15
$_{3½}$	38569.48	10.96
$_{4½}$	38580.44	1.36
$_{5½}$	38581.80	3.93
$_{6½}$	38585.73	− 0.08
$_{7½}$	38585.65	
$153_{4½}$	38661.42	
$154_{3½}$	38676.75	
$155_{3½}$	38738.68	
$156_{5½}$	38852.45	
$\mathrm{y^6P_{3½}}$	38917.62	− 104.69
$_{2½}$	39022.31	− 129.38
$_{1½}$	39151.69	
$\mathrm{f^6S^{\circ}_{2½}}$	*38933.74*	
$157_{4½}$	38975.04	
$\mathrm{y^8F_{6½}}$	39063.29	22.98
$_{5½}$	39040.31	− 11.34
$_{4½}$	39051.65	− 7.92
$_{3½}$	39059.57	− 26.21
$_{2½}$	39085.78	− 43.54
$_{1½}$	39129.32	− 2.78
$_{½}$	39132.10	
$\mathrm{x^6P_{3½}}$	39126.90	− 52.33
$_{2½}$	39179.23	− 99.60
$_{1½}$	39278.83	
$\mathrm{i^8S^{\circ}_{3½}}$	*39242.56*	
$\mathrm{f^{10}D^{\circ}_{2½}}$	*39267.62*	17.11
$_{3½}$	*39284.73*	21.40
$_{4½}$	*39306.13*	26.55
$_{5½}$	*39332.68*	36.51
$_{6½}$	*39369.19*	

TABLE 6—*Continued*

Desig.	Level	Diff.
$158_{5\frac{1}{2},\,6\frac{1}{2}}$	39291.36	
$h^8D^\circ_{5\frac{1}{2}}$	*39486.20*	−5.29
$_{4\frac{1}{2}}$	*39491.49*	−5.07
$_{3\frac{1}{2}}$	*39496.56*	−3.75
$_{2\frac{1}{2}}$	*39500.31*	
$_{1\frac{1}{2}}$	....	
$159_{2\frac{1}{2}}$	40073.57	
$y^6F_{5\frac{1}{2}}$	40198.00	−2.43
$_{4\frac{1}{2}}$	40200.43	−0.71
$_{3\frac{1}{2}}$	40201.14	−2.54
$_{2\frac{1}{2}}$	40203.68	0.31
$_{1\frac{1}{2}}$	40203.37	−3.34
$_{\frac{1}{2}}$	40206.71	
$x^8F_{\frac{1}{2}}$	40217.90	5.76
$_{1\frac{1}{2}}$	40223.66	13.69
$_{2\frac{1}{2}}$	40237.35	−0.71
$_{3\frac{1}{2}}$	40236.64	13.04
$_{4\frac{1}{2}}$	40249.68	−13.54
$_{5\frac{1}{2}}$	40236.14	8.52
$_{6\frac{1}{2}}$	40244.66	
$f^{10}P^\circ_{3\frac{1}{2}}$	*40298.02*	556.09
$_{4\frac{1}{2}}$	*40854.11*	589.59
$_{5\frac{1}{2}}$	*41443.70*	
$160_{4\frac{1}{2},\,5\frac{1}{2}}$	40302.07	
$5^\circ_{3\frac{1}{2}}$	*40576.00*	
$6^\circ_{2\frac{1}{2},\,3\frac{1}{2}}$	*40624.10*	
$7^\circ_{3\frac{1}{2}}$	*40650.51*	
$8^\circ_{4\frac{1}{2},\,5\frac{1}{2}}$	*40764.08*	
$9^\circ_{4\frac{1}{2}}$	*40768.31*	
$161_{4\frac{1}{2},\,5\frac{1}{2}}$	40838.66	
$10^\circ_{3\frac{1}{2},\,4\frac{1}{2}}$	*40993.18*	
$11^\circ_{3\frac{1}{2},\,4\frac{1}{2}}$	*41037.58:*	
$162_{5\frac{1}{2},\,6\frac{1}{2}}$	41054.83:	
$x^{10}F_{7\frac{1}{2}}$	41152.12	−32.00
$_{6\frac{1}{2}}$	41184.12	7.39
$_{5\frac{1}{2}}$	41176.73	5.73
$_{4\frac{1}{2}}$	41171.00	1.80
$_{3\frac{1}{2}}$	41169.20	2.06
$_{2\frac{1}{2}}$	41167.14	1.12
$_{1\frac{1}{2}}$	41166.02	
$f^6D^\circ_{4\frac{1}{2}}$	*41174.74*	−10.64
$_{3\frac{1}{2}}$	*41185.38*	−15.83
$_{2\frac{1}{2}}$	*41201.21*	−7.39
$_{1\frac{1}{2}}$	*41208.60*	
$_{\frac{1}{2}}$	....	
$12^\circ_{4\frac{1}{2},\,5\frac{1}{2}}$	*41378.47*	
$13^\circ_{4\frac{1}{2},\,5\frac{1}{2}}$	*41395.92*	
$14^\circ_{3\frac{1}{2},\,4\frac{1}{2}}$	*41631.37*	
$15^\circ_{4\frac{1}{2},\,5\frac{1}{2}}$	*42721.09:*	
$16^\circ_{3\frac{1}{2},\,4\frac{1}{2}}$	*43100.66*	
$17^\circ_{4\frac{1}{2},\,5\frac{1}{2}}$	*43212.06:*	
18°	*43466.82*	
$19^\circ_{3\frac{1}{2},\,4\frac{1}{2}}$	*43508.06:*	

(5) the wave number, and (6) the classification. Only those lines which have been classified are here given, as the complete list has so recently been published.[4] When two or more predicted positions agree with an observed line (within the limits of resolution appropriate to its intensity), both are given in the table. Combinations which probably contribute but little to the observed intensity are in parentheses.

TABLE 7

CLASSIFIED LINES OF *Eu* I

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
9326.08	+1	4	V	10719.68	$y^8P_{4½}-e^{10}P^{\circ}_{5½}$
9085.33	+1	3	V	11003.74	$b^8D^{\circ}_{1½}-z^8D_{2½}$
9083.06	+1	10	V	11006.49	$b^8D^{\circ}_{1½}-z^8D_{1½}$
9058.00	−1	2	V	11036.94	$b^8D^{\circ}_{2½}-z^8D_{1½}$
9024.33	0	3	V	11078.11	$b^8D^{\circ}_{2½}-z^8D_{3½}$
9018.06	−1	5	V	11085.82	$b^8D^{\circ}_{3½}-z^8D_{2½}$
8982.50	+3	4	V	11129.70	$b^8D^{\circ}_{3½}-z^8D_{3½}$
8965.46	0	4	V	11150.86	$b^8D^{\circ}_{1½}-z^{10}D_{2½}$
8964.3	+6	4	V	11152.3	$b^8D^{\circ}_{4½}-109_{3½}$
8961.69	+6	3	V	11155.55	$z^{10}F_{5½}-9^{\circ}_{4½}$
8935.59	+3	5	V	11188.14	$b^8D^{\circ}_{4½}-111_{5½}$
8934.42	−1	12	V	11189.60	$b^8D^{\circ}_{3½}-z^8D_{4½}$
8917.64	−2	20	V	11210.65	$b^8D^{\circ}_{4½}-z^8D_{3½}$
8899.94	{−2 −27}	8	V	11232.95	{$b^8D^{\circ}_{3½}-z^{10}D_{2½}$ $(z^{10}F_{6½}-f^{10}P^{\circ}_{5½})$
8893.26	+2	2	V	11241.39	$z^{10}F_{5½}-f^{10}P^{\circ}_{4½}$
8883.03	−5	2	V	11254.33	$a^6D^{\circ}_{1½}-z^8D_{2½}$
8880.81	−10	2	V	11257.15	$a^6D^{\circ}_{1½}-z^8D_{1½}$
8870.30	0	40	V	11270.48	$b^8D^{\circ}_{4½}-z^8D_{4½}$
8816.95	−2	4	V	11338.68	$a^6D^{\circ}_{2½}-z^8D_{1½}$
8791.1	−4	6	V	11372.0	$b^8D^{\circ}_{5½}-110_{4½,\ 5½}$
8790.88	−4	25	IV ?	11372.30	$b^8D^{\circ}_{5½}-111_{5½}$
8789.33	+2	3	V	11374.31	$b^8D^{\circ}_{2½}-z^{10}D_{3½}$
8785.06	{0 +5}	4	V	11379.84	{$a^6D^{\circ}_{2½}-z^8D_{3½}$ $(z^{10}F_{3½}-f^{10}P^{\circ}_{3½})$
8782.46	+1	10	IV	11383.21	$b^8D^{\circ}_{4½}-z^8D_{5½}$
8773.30	+3	3	V	11395.09	$b^8D^{\circ}_{3½}-112_{2½}$
8751.66	−8	6	IV ?	11423.27	$z^6P_{3½}-e^{10}S^{\circ}_{4½}$
8749.62	+2	4	V	11425.94	$b^8D^{\circ}_{3½}-z^{10}D_{3½}$
8743.83	−3	10	V	11433.50	$a^6D^{\circ}_{3½}-z^8D_{2½}$
8727.77	+1	30	IV	11454.54	$b^8D^{\circ}_{5½}-z^8D_{4½}$
8710.39	0	2	V	11477.39	$a^6D^{\circ}_{3½}-z^8D_{3½}$
8704.50	−3	6	IV	11485.16	$b^8D^{\circ}_{4½}-113_{3½}$
8665.21	0	4	IV	11537.24	$a^6D^{\circ}_{3½}-z^8D_{4½}$
8658.55	+7	2	V	11546.11	$a^6D^{\circ}_{4½}-111_{5½}$
8642.67	−2	200	III ?	11567.32	$b^8D^{\circ}_{5½}-z^8D_{5½}$
8641.69	+1	6	IV	11568.64	$a^6D^{\circ}_{4½}-z^8D_{3½}$
8632.8	−2	1	V	11580.6	$a^6D^{\circ}_{3½}-z^{10}D_{2½}$
8631.74	+1	4	V	11581.97	$z^{10}F_{4½}-9^{\circ}_{4½}$
8597.19	−1	15	IV	11628.52	$a^6D^{\circ}_{4½}-z^8D_{4½}$
8514.65	0	8	IV A	11741.24	$a^6D^{\circ}_{4½}-z^8D_{5½}$
8507.34	−2	2	V	11751.33	$b^8D^{\circ}_{4½}-z^{10}D_{4½}$
8464.71	−5	40	III A	11810.51	$z^6P_{2½}-e^8S^{\circ}_{3½}$
8450.1	+8	2	....	11830.9	$z^{10}F_{5½}-f^{10}P^{\circ}_{5½}$
8441.5	+10	1	V	11843.0	$a^6D^{\circ}_{4½}-113_{3½}$
8371.80	−2	20	IV	11941.58	$b^8D^{\circ}_{1½}-z^8F_{½}$
8335.56	0	10	IV	11993.50	$b^8D^{\circ}_{1½}-z^8F_{1½}$
8314.47	+1	2?	IV ?	12023.92	$b^8D^{\circ}_{2½}-z^8F_{1½}$
8274.62	0	10	IV	12081.83	$b^8D^{\circ}_{1½}-z^8F_{2½}$
8253.82	−1	15	III	12112.28	$b^8D^{\circ}_{2½}-z^8F_{2½}$

* Unit 0.01 A.

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
8226.81	−1	250	II	12152.04	$z^8P_{4½}-e^{10}S^{\circ}_{4½}$
8212.8	−12	1	....	12172.8	$a^8D^{\circ}_{4½}-100_{3½,\,4½}$?
8209.80	−1	500	II	12177.22	$z^6P_{3½}-e^8S^{\circ}_{3½}$
8188.1	+7	3	....	12209.5	$z^{10}F_{4½}-13^{\circ}_{4½,\,5½}$
8168.09	−1	30	III	12239.40	$b^8D^{\circ}_{2½}-z^8F_{3½}$
8133.75	−4	5	IV	12291.08	$b^8D^{\circ}_{3½}-z^8F_{3½}$
8080.5	−12	3	IV ?	12372.1	$b^8D^{\circ}_{4½}-z^8F_{3½}$
8050.53	−2	6	IV	12418.13	$b^8D^{\circ}_{3½}-115_{4½}$
8015.47	−1	50	III	12472.45	$b^8D^{\circ}_{3½}-z^8F_{4½}$
7998.43	−2	3	IV	12499.02	$b^8D^{\circ}_{4½}-115_{4½}$
7969.02	−3	4	IV	12545.15	$y^8P_{4½}-4^{\circ}$
7963.77	−6	1	V	12553.42	$b^8D^{\circ}_{4½}-z^8F_{4½}$
7890.52	−2	5	IV ?	12669.95	$b^8D^{\circ}_{5½}-z^{10}D_{6½}$
7887.99	−1	500	II	12674.02	$z^6P_{1½}-e^6S^{\circ}_{2½}$
7882.34	−2	30	III	12683.10	$b^8D^{\circ}_{5½}-115_{4½}$
7874.68	+2	3	IV ?	12695.44	$b^8D^{\circ}_{4½}-116_{4½,\,5½}$
7848.69	−5	8	IV A	12737.48	$b^8D^{\circ}_{5½}-z^8F_{4½}$
7830.60	+1	3	IV ?	12766.90	$b^8D^{\circ}_{4½}-y^{10}P_{3½}$
7818.21	−1	40	III	12787.14	$b^8D^{\circ}_{4½}-z^8F_{5½}$
7803.32	−2	50	III A	12811.54	$z^8P_{3½}-e^{10}S^{\circ}_{4½}$
7759.36	−2	4	IV	12884.12	$b^8D^{\circ}_{3½}-y^{10}P_{4½}$
7746.19	+3	500	II	12906.02	$z^8P_{4½}-e^8S^{\circ}_{3½}$
7742.57	+1	300	II	12912.06	$z^6P_{2½}-e^6S^{\circ}_{2½}$
7710.92	−5	4	IV A	12965.06	$b^8D^{\circ}_{4½}-y^{10}P_{4½}$
7708.52	−7	1	V	12969.09	$b^8D^{\circ}_{3½}-118_{3½}$
7707.21	−7	1	V	12971.30	$b^8D^{\circ}_{5½}-z^8F_{5½}$
7627.65	−2	8	IV A	13106.60	$a^8D^{\circ}_{5½}-z^{10}F_{4½}$
7603.87	−2	6	IV A	13147.58	$a^8D^{\circ}_{4½}-101_{3½}$
7603.05	+2	1	IV	13149.00	$b^8D^{\circ}_{5½}-y^{10}P_{4½}$
7583.91	+3	1000	II	13182.19	$z^{10}P_{5½}-e^{10}S^{\circ}_{4½}$
7552.00	−3	8	III A	13237.89	$a^8D^{\circ}_{4½}-z^{10}F_{3½}$
7547.32	−1	40	II A	13246.10	$a^8D^{\circ}_{3½}-z^{10}F_{2½}$
7533.02	−2	30	II A	13271.24	$a^8D^{\circ}_{2½}-z^{10}F_{1½}$
7532.17	−1	4	IV	13272.74	$y^8P_{4½}-e^8D^{\circ}_{3½}$
7528.70	−1	400	II	13278.85	$z^6P_{3½}-e^6S^{\circ}_{2½}$
7521.25	−2	8	IV	13292.01	$y^8P_{4½}-e^8P^{\circ}_{3½}$
7508.50	−2	20	III A	13314.58	$b^8D^{\circ}_{5½}-z^8F_{6½}$
7507.02	−3	8	IV	13317.20	$b^8D^{\circ}_{4½}-y^{10}P_{5½}$
7491.00	0	30	IV	13345.68	$y^8P_{4½}-e^8D^{\circ}_{4½}$
7481.52	−4	2	IV	13362.60	$y^8P_{3½}-e^8D^{\circ}_{2½}$
7470.53	0	50	II A	13382.25	$a^8D^{\circ}_{1½}-z^{10}F_{1½}$
7450.34	+3	5	III A	13418.51	$a^8D^{\circ}_{2½}-z^{10}F_{2½}$
7444.62	−1	15	IV	13428.82	$y^8P_{3½}-e^8D^{\circ}_{3½}$
7436.59	−1	200	III	13443.33	$y^8P_{4½}-e^8D^{\circ}_{5½}$
7404.70	0	15	IV	13501.22	$b^8D^{\circ}_{5½}-y^{10}P_{5½}$
7404.41	+1	40	III	13501.75	$y^8P_{3½}-e^8D^{\circ}_{4½}$
7402.06	0	4	IV A	13506.04	$a^8D^{\circ}_{4½}-z^{10}F_{4½}$
7399.96	+1	4	IV A	13509.87	$z^6P_{2½}-f^8S^{\circ}_{3½}$
7394.86	−1	12	IV	13519.19	$y^8P_{4½}-e^8P^{\circ}_{4½}$
7392.70	−2	5	IV	13523.14	$y^8P_{2½}-e^8D^{\circ}_{2½}$
7389.16	−1	150	II	13529.62	$a^8D^{\circ}_{1½}-z^{10}F_{2½}$
7387.36	+1	20	III A	13532.91	$a^8D^{\circ}_{5½}-z^{10}F_{5½}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
7369.60	+1	600	II	13565.53	$z^8P_{3\frac{1}{2}}-e^8S^{\circ}_{3\frac{1}{2}}$
7367.70	+1	10	IV	13569.02	$y^8P_{3\frac{1}{2}}-e^8P^{\circ}_{2\frac{1}{2}}$
7362.25	0	80	II	13579.07	$a^8D^{\circ}_{2\frac{1}{2}}-101_{3\frac{1}{2}}$
7356.65	−1	25	III	13589.41	$y^8P_{2\frac{1}{2}}-e^8D^{\circ}_{3\frac{1}{2}}$
7346.25	+1	20	III	13608.64	$y^8P_{2\frac{1}{2}}-e^8P^{\circ}_{3\frac{1}{2}}$
7341.9	−1	1−	V	13616.7	$y^8P_{4\frac{1}{2}}-f^8D^{\circ}_{3\frac{1}{2}}$
7336.18	+1	800	II	13627.32	$z^8P_{2\frac{1}{2}}-e^8S^{\circ}_{3\frac{1}{2}}$
7313.63	+1	100	II A	13669.34	$a^8D^{\circ}_{2\frac{1}{2}}-z^{10}F_{3\frac{1}{2}}$
7310.46	−1	40	III	13675.27	$y^8P_{3\frac{1}{2}}-e^8P^{\circ}_{4\frac{1}{2}}$
7297.56	+2	20	IV	13699.44	$y^8P_{4\frac{1}{2}}-f^8D^{\circ}_{5\frac{1}{2}}$
7281.53	+2	40	III	13729.59	$y^8P_{2\frac{1}{2}}-e^8P^{\circ}_{2\frac{1}{2}}$
7262.77	+1	200	II	13765.06	$a^8D^{\circ}_{3\frac{1}{2}}-z^{10}F_{4\frac{1}{2}}$
7258.72	+2	80	III	13772.74	$y^8P_{3\frac{1}{2}}-f^8D^{\circ}_{3\frac{1}{2}}$
7248.08	+2	20	IV	13792.96	$y^8P_{3\frac{1}{2}}-f^8D^{\circ}_{2\frac{1}{2}}$
7224.68	+1	100	III	13837.63	$y^8P_{2\frac{1}{2}}-f^8D^{\circ}_{1\frac{1}{2}}$
7204.41	+4	15?	III A	13876.57	$z^6P_{3\frac{1}{2}}-f^8S^{\circ}_{3\frac{1}{2}}$
7175.55	+2	200	II	13932.38	$a^8D^{\circ}_{4\frac{1}{2}}-z^{10}F_{5\frac{1}{2}}$
7175.0	−2	15	IV	13933.4	$y^8P_{2\frac{1}{2}}-f^8D^{\circ}_{3\frac{1}{2}}$
7164.66	+1	40	III	13953.56	$y^8P_{2\frac{1}{2}}-f^8D^{\circ}_{2\frac{1}{2}}$
7129.33	−1	4	IV	14022.70	$b^8D^{\circ}_{2\frac{1}{2}}-x^8P_{2\frac{1}{2}}$
7114.48	+12	6	{V V E}	14051.97	$(y^8P_{4\frac{1}{2}}-f^8D^{\circ}_{4\frac{1}{2}})$
7106.48	+4	1500	I	14067.79	$a^8S^{\circ}_{3\frac{1}{2}}-z^{10}P_{3\frac{1}{2}}$
7074.54	+1	150	II A	14131.30	$a^8D^{\circ}_{5\frac{1}{2}}-z^{10}F_{6\frac{1}{2}}$
7040.20	0	2500	II	14200.24	$z^{10}P_{4\frac{1}{2}}-e^{10}S^{\circ}_{4\frac{1}{2}}$
7036.23	+2	3	IV	14208.25	$y^8P_{3\frac{1}{2}}-f^8D^{\circ}_{4\frac{1}{2}}$
6973.34	0	8	IV A	14336.38	$y^8P_{4\frac{1}{2}}-g^8D^{\circ}_{3\frac{1}{2}}$
6947.49	−4	2	IV	14389.72	$b^8D^{\circ}_{3\frac{1}{2}}-x^8P_{3\frac{1}{2}}$
6934.3	−2	2	V	14417.1	$a^6D^{\circ}_{2\frac{1}{2}}-119_{3\frac{1}{2}}$
6914.82	+1	150	III	14457.71	$y^8P_{4\frac{1}{2}}-g^8D^{\circ}_{4\frac{1}{2}}$
6910.17	0	50	III	14467.44	$y^8P_{3\frac{1}{2}}-g^8D^{\circ}_{2\frac{1}{2}}$
6908.71	+1	10	IV A	14470.50	$b^8D^{\circ}_{4\frac{1}{2}}-x^8P_{3\frac{1}{2}}$
6903.67	+1	1000	III	14481.06	$y^8P_{4\frac{1}{2}}-g^8D^{\circ}_{5\frac{1}{2}}$
6898.21	−2	150	III	14492.52	$y^8P_{3\frac{1}{2}}-g^8D^{\circ}_{3\frac{1}{2}}$
6890.75	−2	3	IV A	14508.21	$z^6P_{3\frac{1}{2}}-e^{10}P^{\circ}_{3\frac{1}{2}}$
6864.54	−2	3000	I	14563.61	$a^8S^{\circ}_{3\frac{1}{2}}-z^{10}P_{4\frac{1}{2}}$
6847.04	−1	100	III	14600.83	$y^8P_{2\frac{1}{2}}-g^8D^{\circ}_{1\frac{1}{2}}$
6844.83	−2	200	III A	14605.55	$z^8P_{4\frac{1}{2}}-f^8S^{\circ}_{3\frac{1}{2}}$
6840.93	−2	200	III	14613.87	$y^8P_{3\frac{1}{2}}-g^8D^{\circ}_{4\frac{1}{2}}$
6834.30	0	100	III	14628.05	$y^8P_{2\frac{1}{2}}-g^8D^{\circ}_{2\frac{1}{2}}$
6830.7	−13	1	IV ?	14635.8	$a^6D^{\circ}_{4\frac{1}{2}}-120_{3\frac{1}{2}}?$
6822.61	−2	20	IV	14653.11	$y^8P_{2\frac{1}{2}}-g^8D^{\circ}_{3\frac{1}{2}}$
6816.06	0	800	III A	14667.19	$z^8P_{3\frac{1}{2}}-e^6S^{\circ}_{2\frac{1}{2}}$
6802.72	0	2500	II	14695.95	$z^{10}P_{3\frac{1}{2}}-e^{10}S^{\circ}_{4\frac{1}{2}}$
6787.48	0	300	III A	14728.95	$z^8P_{2\frac{1}{2}}-e^6S^{\circ}_{2\frac{1}{2}}$
6784.87†	+11	4	IV	14734.62	$b^8D^{\circ}_{4\frac{1}{2}}-w^8P_{4\frac{1}{2}}?$
6782.7	+4	20?	III A	14739.3	$a^8D^{\circ}_{5\frac{1}{2}}-110_{4\frac{1}{2},\,5\frac{1}{2}}$
6782.54	0	600	III A	14739.68	$a^8D^{\circ}_{5\frac{1}{2}}-111_{5\frac{1}{2}}$
6772.44	−2	8	IV	14761.66	$b^8D^{\circ}_{1\frac{1}{2}}-w^8P_{2\frac{1}{2}}$
6758.53	0	8	IV	14792.04	$b^8D^{\circ}_{2\frac{1}{2}}-w^8P_{2\frac{1}{2}}$
6755.82	−1	2	IV	14797.97	$b^8D^{\circ}_{2\frac{1}{2}}-w^8P_{3\frac{1}{2}}$

† Wide line—King.

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
6744.88	0	600	II	14821.98	$a^8D^\circ_{5\frac12}-z^8D_{4\frac12}$
6741.9	+2	2	IV A	14828.5	$a^6D^\circ_{4\frac12}-x^8P_{3\frac12}$
6735.0	−1	3	IV	14843.7	$b^8D^\circ_{3\frac12}-w^8P_{2\frac12}$
6732.36	+2	4	IV	14849.54	$b^8D^\circ_{3\frac12}-w^8P_{3\frac12}$
6710.45	+1	30	III	14898.03	$y^8P_{4\frac12}-g^8S^\circ_{3\frac12}$
6701.06	+2	20	III A	14918.90	$b^8D^\circ_{5\frac12}-w^8P_{4\frac12}$
6695.84	−3	4	IV A	14930.53	$b^8D^\circ_{4\frac12}-w^8P_{3\frac12}$
6693.96	0	1500	II	14934.72	$a^8D^\circ_{5\frac12}-z^8D_{5\frac12}$
6685.21	+1	400	II	14954.27	$z^{10}P_{4\frac12}-e^8S^\circ_{3\frac12}$
6682.00	−1	15	IV	14961.5	$y^8P_{3\frac12}-e^6D^\circ_{4\frac12}$
6681.6	−6	1	IV A	14962.4	$a^8D^\circ_{4\frac12}-108_{3\frac12,\,4\frac12}$
6671.89	−1	10	IV	14984.13	$y^8P_{3\frac12}-e^6D^\circ_{3\frac12}$
6663.3	−7	1	IV	15003.4	$y^8P_{3\frac12}-e^6D^\circ_{2\frac12}$
6640.8	−7	3	IV	15054.3	$y^8P_{3\frac12}-g^8S^\circ_{3\frac12}$
6623.72	−9	3	V E IV	15093.10	$a^6D^\circ_{4\frac12}-w^8P_{4\frac12}$
6623.3	−14 +15	1	IV A	15094.1	$a^6D^\circ_{2\frac12}-w^8P_{2\frac12}$? $b^8D^\circ_{4\frac12}-x^8P_{4\frac12}$?
6621.22	+4	3	IV A	15098.79	$b^8D^\circ_{5\frac12}-x^{10}P_{4\frac12}$
6619.19	−3	6	III A	15103.42	$a^8D^\circ_{4\frac12}-109_{3\frac12}$
6603.7	−2	10?	III	15138.9	$a^8D^\circ_{4\frac12}-110_{4\frac12,\,5\frac12}$
6603.55	−2	200	III A	15139.20	$a^8D^\circ_{4\frac12}-111_{5\frac12}$
6601.1	−4	3	IV	15144.8	$y^8P_{2\frac12}-e^6D^\circ_{3\frac12}$
6593.79	0	400	II	15161.60	$a^8D^\circ_{4\frac12}-z^8D_{3\frac12}$
6592.9	+6	3	IV	15163.7	$y^8P_{2\frac12}-e^6D^\circ_{2\frac12}$
6587.0	+10	2	IV	15177.2	$y^8P_{2\frac12}-e^6D^\circ_{1\frac12}$
6581.0	+10	1	IV A	15191.1	$a^6D^\circ_{3\frac12}-w^8P_{2\frac12}$
657[illegible].3	−3	2	IV A	15197.3	$a^6D^\circ_{3\frac12}-w^8P_{3\frac12}$
6570.76	0	20	III	15214.74	$y^8P_{2\frac12}-g^8S^\circ_{3\frac12}$
6567.87	+1	600	II	15221.44	$a^8D^\circ_{4\frac12}-z^8D_{4\frac12}$
6561.17	+2	20	III A	15236.98	$z^8P_{4\frac12}-e^{10}P^\circ_{3\frac12}$
6549.12	−1	60	III	15265.02	$z^8P_{3\frac12}-f^8S^\circ_{3\frac12}$
6543.2	−12	2	IV	15278.8	$b^8D^\circ_{5\frac12}-x^8P_{4\frac12}$
6539.02	−5	2	IV A	15288.60	$a^6D^\circ_{4\frac12}-w^8P_{3\frac12}$
6532.96	+1	6	III A	15302.78	$a^8D^\circ_{5\frac12}-z^{10}D_{4\frac12}$
6522.72	−1	80	III	15326.80	$z^8P_{2\frac12}-f^8S^\circ_{3\frac12}$
6519.59	+2	· 600	II	15334.16	$a^8D^\circ_{4\frac12}-z^8D_{5\frac12}$
6507.60	−1	3	IV A	15362.41	$a^8D^\circ_{3\frac12}-109_{3\frac12}$
6501.55	0	300	II	15376.71	$a^8D^\circ_{3\frac12}-z^8D_{2\frac12}$
6483.02	−1	100	II	15420.66	$a^8D^\circ_{3\frac12}-z^8D_{3\frac12}$
6476.55	+2	20	III A	15436.06	$a^8D^\circ_{4\frac12}-113_{3\frac12}$
6470.70	−1	80	II A	15450.02	$z^{10}P_{3\frac12}-e^8S^\circ_{3\frac12}$
6469.65	−2	1	IV A	15452.52	$a^6D^\circ_{4\frac12}-x^8P_{4\frac12}$
6467.44	+1	10	III A	15457.81	$a^8D^\circ_{4\frac12}-z^{10}D_{3\frac12}$
6457.96	0	600	II	15480.50	$a^8D^\circ_{3\frac12}-z^8D_{4\frac12}$
6439.93	−1	60	II	15523.84	$a^8D^\circ_{3\frac12}-z^{10}D_{2\frac12}$
6435.35	0	8	III A	15534.89	$a^8D^\circ_{2\frac12}-109_{3\frac12}$
6429.44	+2	2	III A	15549.16	$a^8D^\circ_{2\frac12}-z^8D_{2\frac12}$
6428.29	0	300	II	15551.95	$a^8D^\circ_{2\frac12}-z^8D_{1\frac12}$
6413.19	−1	1	IV A	15588.56	$a^{10}D^\circ_{5\frac12}-102_{4\frac12}$
6411.32	+1	600	II	15593.11	$a^8D^\circ_{2\frac12}-z^8D_{3\frac12}$
6410.04	+1	1200	II	15596.22	$a^{10}D^\circ_{2\frac12}-z^{10}F_{1\frac12}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
6406.11	0	200	II	15605.79	$a^{10}D^{\circ}_{4\frac{1}{2}} - 101_{3\frac{1}{2}}$
6400.93	0	1000	II	15618.42	$a^{10}D^{\circ}_{3\frac{1}{2}} - z^{10}F_{2\frac{1}{2}}$
6389.58	0	15	III A	15646.16	$a^{8}D^{\circ}_{5\frac{1}{2}} - z^{10}D_{5\frac{1}{2}}$
6383.86	+ 2	500	II	15660.18	$a^{8}D^{\circ}_{1\frac{1}{2}} - z^{8}D_{2\frac{1}{2}}$
6382.73	+ 1	300	II	15662.96	$a^{8}D^{\circ}_{1\frac{1}{2}} - z^{8}D_{1\frac{1}{2}}$
6373.34	0	10	III A	15686.03	$a^{8}D^{\circ}_{3\frac{1}{2}} - 112_{2\frac{1}{2}}$
6369.6	− 3	?	IV A	15695.2	$a^{8}D^{\circ}_{3\frac{1}{2}} - 113_{3\frac{1}{2}}$
6369.25	− 1 + 9	400	II	15696.10	$a^{10}D^{\circ}_{4\frac{1}{2}} - z^{10}F_{3\frac{1}{2}}$ $(a^{8}D^{\circ}_{2\frac{1}{2}} - z^{10}D_{2\frac{1}{2}})$
6366.76	+ 2	4	III A	15702.24	$a^{8}D^{\circ}_{4\frac{1}{2}} - z^{10}D_{4\frac{1}{2}}$
6360.83	− 1	1	IV A	15716.88	$a^{8}D^{\circ}_{3\frac{1}{2}} - z^{10}D_{3\frac{1}{2}}$
6355.89	0	300	II	15729.10	$a^{10}D^{\circ}_{5\frac{1}{2}} - z^{10}F_{4\frac{1}{2}}$
6350.04	+ 1	1000	II	15743.59	$a^{10}D^{\circ}_{2\frac{1}{2}} - z^{10}F_{2\frac{1}{2}}$
6335.82	0	400	II	15778.92	$a^{10}D^{\circ}_{3\frac{1}{2}} - 101_{3\frac{1}{2}}$
6324.42	0	40	IV A	15807.36	$a^{8}D^{\circ}_{1\frac{1}{2}} - z^{10}D_{2\frac{1}{2}}$
6318.58	0	30	III	15821.97	$b^{8}D^{\circ}_{4\frac{1}{2}} - z^{6}F_{5\frac{1}{2}}$
6317.87	− 1 − 2	3	IV A	15823.75	$y^{8}P_{4\frac{1}{2}} - e^{6}P^{\circ}_{3\frac{1}{2}}$ $a^{10}D^{\circ}_{4\frac{1}{2}} - 102_{4\frac{1}{2}}$
6313.78	0	100	III A	15834.00	$a^{10}D^{\circ}_{6\frac{1}{2}} - z^{10}F_{5\frac{1}{2}}$
6308.1	+12	2	IV	15848.3	$b^{8}D^{\circ}_{2\frac{1}{2}} - 123_{3\frac{1}{2}}$
6304.00	− 3	[5]‡	IV A	15858.6	$a^{8}D^{\circ}_{2\frac{1}{2}} - 112_{2\frac{1}{2}}$
6300.42	+ 2	20	III A	15867.58	$a^{8}D^{\circ}_{2\frac{1}{2}} - 113_{3\frac{1}{2}}$
6299.77	0 − 2	800	II	15869.22	$a^{10}D^{\circ}_{3\frac{1}{2}} - z^{10}F_{3\frac{1}{2}}$ $(z^{8}P_{4\frac{1}{2}} - e^{10}P^{\circ}_{5\frac{1}{2}})$
6298.08	0	2	IV	15873.47	$a^{8}D^{\circ}_{4\frac{1}{2}} - 114_{4\frac{1}{2}}$
6291.79	0	15	III A	15889.34	$a^{8}D^{\circ}_{2\frac{1}{2}} - z^{10}D_{3\frac{1}{2}}$
6291.34	+ 2	300	I A	15890.48	$a^{8}S^{\circ}_{3\frac{1}{2}} - z^{8}P_{2\frac{1}{2}}$
6288.95	− 1	30	III	15896.52	$z^{8}P_{3\frac{1}{2}} - e^{10}P^{\circ}_{3\frac{1}{2}}$
6287.45	− 4	1	IV A	15900.31	$b^{8}D^{\circ}_{3\frac{1}{2}} - 123_{3\frac{1}{2}}$
6285.95	0	80	III A	15904.11	$a^{10}D^{\circ}_{2\frac{1}{2}} - 101_{3\frac{1}{2}}$
6283.87	− 2	1	IV A	15909.37	$b^{8}D^{\circ}_{1\frac{1}{2}} - 124_{2\frac{1}{2}}$
6266.95	0	150	I A	15952.32	$a^{8}S^{\circ}_{3\frac{1}{2}} - z^{8}P_{3\frac{1}{2}}$
6264.60	− 1	15	III A	15958.31	$z^{8}P_{2\frac{1}{2}} - e^{10}P^{\circ}_{3\frac{1}{2}}$
6263.42	0	20	III A	15961.31	$a^{8}D^{\circ}_{3\frac{1}{2}} - z^{10}D_{4\frac{1}{2}}$
6262.25	0	1500	II	15964.29	$a^{10}D^{\circ}_{4\frac{1}{2}} - z^{10}F_{4\frac{1}{2}}$
6260.16	− 3	4	III A	15969.63	$a^{8}D^{\circ}_{1\frac{1}{2}} - 112_{2\frac{1}{2}}$
6255.65	− 1	1	IV A	15981.14	$b^{8}D^{\circ}_{4\frac{1}{2}} - 123_{3\frac{1}{2}}$
6250.47	+ 1	150	II	15994.38	$a^{10}D^{\circ}_{2\frac{1}{2}} - z^{10}F_{3\frac{1}{2}}$
6249.51	0	1−	IV A	15996.84	$a^{10}D^{\circ}_{3\frac{1}{2}} - 102_{4\frac{1}{2}}$
6245.91	− 1	10	III A	16006.06	$b^{8}D^{\circ}_{5\frac{1}{2}} - z^{6}F_{5\frac{1}{2}}$
6240.71	0	8	III	16019.40	$b^{8}D^{\circ}_{3\frac{1}{2}} - z^{6}F_{4\frac{1}{2}}$
6233.73	0	125	III	16037.33	$a^{8}D^{\circ}_{5\frac{1}{2}} - z^{10}D_{6\frac{1}{2}}$
6230.51	+ 1	40	III A	16045.62	$a^{8}D^{\circ}_{4\frac{1}{2}} - z^{10}D_{5\frac{1}{2}}$
6218.80	+ 2	1	IV A	16075.83	$a^{10}D^{\circ}_{3\frac{1}{2}} - 103_{2\frac{1}{2},\,3\frac{1}{2}}$
6209.35	0	10	III	16100.30	$b^{8}D^{\circ}_{4\frac{1}{2}} - z^{6}F_{4\frac{1}{2}}$
6207.60	0	50	III A	16104.84	$a^{8}D^{\circ}_{5\frac{1}{2}} - z^{8}F_{4\frac{1}{2}}$
6201.65	− 2	2	IV	16120.29	$z^{6}P_{1\frac{1}{2}} - 1^{\circ}_{2\frac{1}{2}}$
6196.94	− 1	2	IV A	16132.54	$a^{8}D^{\circ}_{3\frac{1}{2}} - 114_{4\frac{1}{2}}$
6195.74	0	1−	IV A	16135.67	$z^{6}P_{1\frac{1}{2}} - 2^{\circ}_{1\frac{1}{2},\,2\frac{1}{2}}$
6195.07	0	600	II	16137.41	$a^{10}D^{\circ}_{3\frac{1}{2}} - z^{10}F_{4\frac{1}{2}}$
6193.90	− 2	2	IV A	16140.46	$y^{8}P_{2\frac{1}{2}} - e^{6}P^{\circ}_{3\frac{1}{2}}$

‡ Brackets denote furnace intensity.

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
6191.27	−1	5	IV	16147.32	$b^8D^\circ_{1½}-127_{1½,\,2½}$
6190.2	−3	2	IV	16150.4	$a^6D^\circ_{2½}-123_{3½}$
6188.13	−1	1500	II	16155.51	$a^{10}D^\circ_{5½}-z^{10}F_{5½}$
6186.45	−2	2	IV A	16159.90	$a^6D^\circ_{1½}-124_{2½}$
6185.27	−1	5	IV	16162.98	$a^6D^\circ_{½}-125_{1½}$
6179.63	0	2	IV A	16177.73	$b^8D^\circ_{2½}-127_{1½,\,2½}$
6178.76	−1	150	III	16180.01	$a^6D^\circ_{4½}-z^6F_{5½}$
6171.17	+1	2	IV A	16199.91	$a^6D^\circ_{½}-126_{1½}$
6170.72	−1	1−	IV A	16201.09	$a^{10}D^\circ_{2½}-103_{2½,\,3½}$
6164.14	+2	4	IV	16218.39	$a^6D^\circ_{1½}-125_{1½}$
6158.71	+2	10	IV	16232.68	$y^8P_{4½}-h^8S^\circ_{3½}$
6155.36	−1	15	IV	16241.52	$a^6D^\circ_{2½}-124_{2½}$
6153.27	−2	40	III A	16247.03	$a^8D^\circ_{5½}-116_{4½,\,5½}$
6152.94	−2	2	IV A	16247.91	$a^6D^\circ_{3½}-123_{3½}$
6150.11	+1	4	IV A	16255.38	$a^6D^\circ_{1½}-126_{1½}$
6149.28	0	5	IV A	16257.58	$z^8P_{3½}-e^{10}P^\circ_{4½}$
6149.00	+2	1	IV A	16258.32	$b^8D^\circ_{1½}-128_{2½}$
6139.15	−1	6	IV A	16284.40	$b^8D^\circ_{5½}-z^6F_{4½}$
6133.26	+1	1	IV A	16300.04	$a^6D^\circ_{2½}-125_{1½}$
6126.82	−2	3	IV	16317.17	$b^8D^\circ_{2½}-z^6F_{3½}$
6124.67	0	150	II	16322.90	$a^8D^\circ_{4½}-z^8F_{3½}$
6119.35	−2	2	IV A	16337.09	$a^6D^\circ_{2½}-126_{1½}$
6118.78	0	400	II	16338.61	$a^8D^\circ_{5½}-z^8F_{5½}$
6118.11	+2	6	IV A	16340.40	$b^8D^\circ_{3½}-128_{2½}$
6111.00	+5	3	IV	16359.42	$b^8D^\circ_{3½}-129_{3½,\,4½}$
6108.15	0	150	III	16367.05	$a^6D^\circ_{3½}-z^6F_{4½}$
6107.51	0	15	IV	16368.76	$b^8D^\circ_{3½}-z^6F_{3½}$
6105.67	+1	1	IV A	16373.70	$z^6P_{2½}-2^\circ_{1½,\,2½}$
6102.81	0	2	IV	16381.37	$a^{10}D^\circ_{5½}-105_{4½}$
6100.05	+1	12	IV	16388.78	$y^8P_{3½}-h^8S^\circ_{3½}$
6099.35	0	1200	II	16390.66	$a^{10}D^\circ_{4½}-z^{10}F_{5½}$
6083.84	−1 −10	1200	II	16432.45	$a^{10}D^\circ_{6½}-z^{10}F_{6½}$ $(y^8P_{3½}-e^6P^\circ_{2½})$
6080.94	+6	6	IV	16440.28	$b^8D^\circ_{4½}-129_{3½,\,4½}$?
6077.38	+2 −10	100	III	16449.91	$a^8D^\circ_{4½}-115_{4½}$ $(b^8D^\circ_{4½}-z^6F_{3½})$
6075.58	0	300	II	16454.79	$a^8D^\circ_{3½}-z^8F_{2½}$
6074.30	+2	10	IV	16458.26	$a^6D^\circ_{4½}-z^6F_{4½}$
6062.3	−10	1	IV	16490.8	$b^8D^\circ_{1½}-z^6F_{2½}$
6057.36	+1	600	II	16504.28	$a^8D^\circ_{4½}-z^8F_{4½}$
6052.89	0	8§	III A	16516.47	$a^8D^\circ_{5½}-y^{10}P_{4½}$
6052.23	−1	5	IV A	16518.27	$a^8D^\circ_{5½}-117_{5½,\,6½}$
6051.28	+4	6	IV	16520.86	$b^8D^\circ_{2½}-z^6F_{2½}$
6044.66	0	250	II	16538.96	$a^8D^\circ_{2½}-z^8F_{1½}$
6040.87	+3	8	IV	16549.33	$y^8P_{2½}-h^8S^\circ_{3½}$
6040.05	+2	1	IV A	16551.58	$z^{10}P_{3½}-e^6S^\circ_{2½}$
6033.08	+1	1	IV	16570.70	$b^8D^\circ_{2½}-131_{1½}$
6032.42	+3	10	IV	16572.52	$b^8D^\circ_{3½}-z^6F_{2½}$
6029.00	0	600	II	16581.92	$a^8D^\circ_{3½}-z^8F_{3½}$
6027.08	0	1	IV A	16587.20	$a^{10}D^\circ_{4½}-104_{3½\ \ 4½}$
6025.11	+5	2	IV	16592.62	$y^8P_{2½}-e^6P^\circ_{2½}$

§ Blend with *Eu* II.

TABLE 7—*Continued*

λ	λΔ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
6023.15	+ 1	250	II	16598.02	$a^8D^\circ_{1\frac{1}{2}}-z^8F_{\frac{1}{2}}$
6018.15	− 1	2500	I	16611.81	$a^8S^\circ_{3\frac{1}{2}}-z^8P_{4\frac{1}{2}}$
6016.40	+ 2	4	III A	16616.64	$b^8D^\circ_{1\frac{1}{2}}-v^8P_{2\frac{1}{2}}$
6016.07	+ 4	8	III	16617.56	$b^8D^\circ_{2\frac{1}{2}}-v^8P_{3\frac{1}{2}}$
6015.58	− 2	150	III	16618.91	$a^6D^\circ_{2\frac{1}{2}}-z^6F_{3\frac{1}{2}}$
6012.56	+ 1	400	II	16627.26	$a^8D^\circ_{2\frac{1}{2}}-z^8F_{2\frac{1}{2}}$
6012.20	+ 2	300	II	16628.25	$z^{10}P_{5\frac{1}{2}}-e^{10}P^\circ_{4\frac{1}{2}}$
6005.8	− 7	15	III	16646.0	$a^6D^\circ_{2\frac{1}{2}}-130_{2\frac{1}{2},\,3\frac{1}{2}}$
6005.61	− 2	60	III	16646.50	$a^8D^\circ_{4\frac{1}{2}}-116_{4\frac{1}{2},\,5\frac{1}{2}}$
6004.36	+ 1	300	II	16649.96	$a^8D^\circ_{1\frac{1}{2}}-z^8F_{1\frac{1}{2}}$
6003.02	+ 2	50	III A	16653.68	$z^{10}P_{4\frac{1}{2}}-f^8S^\circ_{3\frac{1}{2}}$
5994.31	+ 4	1	IV A	16677.88	$b^8D^\circ_{2\frac{1}{2}}-u^8P_{2\frac{1}{2}}$
5992.83	− 1	1500	II	16682.00	$a^8D^\circ_{5\frac{1}{2}}-z^8F_{6\frac{1}{2}}$
5990.65	+ 1	1	IV	16688.07	$a^6D^\circ_{3\frac{1}{2}}-128_{2\frac{1}{2}}$
5986.84	+ 3	8	IV A	16698.69	$b^8D^\circ_{3\frac{1}{2}}-v^8P_{2\frac{1}{2}}$
5983.78	− 1 + 8	60	III	16707.23	$a^6D^\circ_{3\frac{1}{2}}-129_{3\frac{1}{2},\,4\frac{1}{2}}$ $(b^8D^\circ_{1\frac{1}{2}}-z^6F_{1\frac{1}{2}})$
5983.14	0	80	III A	16709.01	$a^8D^\circ_{3\frac{1}{2}}-115_{4\frac{1}{2}}$
5980.47	− 3	30	III	16716.47	$a^6D^\circ_{3\frac{1}{2}}-z^6F_{3\frac{1}{2}}$
5979.99	+ 3	6	III A	16717.82	$a^8D^\circ_{4\frac{1}{2}}-y^{10}P_{3\frac{1}{2}}$
5977.41	+ 2	2	IV A	16725.03	$z^6P_{3\frac{1}{2}}-1^\circ_{2\frac{1}{2}}$
5975.80	+ 3	1	IV A	16729.54	$b^8D^\circ_{3\frac{1}{2}}-u^8P_{2\frac{1}{2}}$
5973.71	− 1	5	IV	16735.39	$a^6D^\circ_{\frac{1}{2}}-131_{1\frac{1}{2}}$
5972.75	+ 1 − 8 + 9	800	II	16738.08	$a^8D^\circ_{4\frac{1}{2}}-z^8F_{5\frac{1}{2}}$ $(b^8D^\circ_{2\frac{1}{2}}-z^6F_{1\frac{1}{2}})$ $(a^8D^\circ_{1\frac{1}{2}}-z^8F_{2\frac{1}{2}})$
5971.69	0	60	III	16741.05	$a^6D^\circ_{1\frac{1}{2}}-z^6F_{2\frac{1}{2}}$
5970.87	0	6	IV	16743.35	$a^6D^\circ_{3\frac{1}{2}}-130_{2\frac{1}{2},\,3\frac{1}{2}}$
5968.43	− 1	40	III	16750.20	$b^8D^\circ_{4\frac{1}{2}}-v^8P_{3\frac{1}{2}}$
5967.10	− 2 +17	2500	II	16753.93	$a^{10}D^\circ_{5\frac{1}{2}}-z^{10}F_{6\frac{1}{2}}$ $(a^8D^\circ_{2\frac{1}{2}}-z^8F_{3\frac{1}{2}})$
5964.85	+ 3 + 5	1	IV	16760.25	$a^{10}D^\circ_{3\frac{1}{2}}-104_{3\frac{1}{2},\,4\frac{1}{2}}$ $a^{10}D^\circ_{4\frac{1}{2}}-106_{3\frac{1}{2},\,4\frac{1}{2}}$?
5963.76	+ 1	400	II	16763.31	$a^8D^\circ_{3\frac{1}{2}}-z^8F_{4\frac{1}{2}}$
5955.75	− 3	8	IV A	16785.86	$z^6P_{3\frac{1}{2}}-3^\circ_{3\frac{1}{2},\,4\frac{1}{2}}$
5954.28	+ 1	60	III A	16790.00	$b^8D^\circ_{1\frac{1}{2}}-y^8D_{2\frac{1}{2}}$
5953.97	− 1	80§	IV	16790.87	$a^6D^\circ_{1\frac{1}{2}}-131_{1\frac{1}{2}}$
5953.49	− 3 + 4 − 2	60	III	16792.23	$b^8D^\circ_{1\frac{1}{2}}-y^8D_{1\frac{1}{2}}$ $(b^8D^\circ_{3\frac{1}{2}}-u^8P_{4\frac{1}{2}})$ $(b^8D^\circ_{1\frac{1}{2}}-z^6F_{\frac{1}{2}})$
5951.22	− 6	2	IV	16798.63	$a^6D^\circ_{4\frac{1}{2}}-129_{3\frac{1}{2},\,4\frac{1}{2}}$
5950.37	+ 2	2	IV	16801.03	$y^8P_{2\frac{1}{2}}-e^6P^\circ_{1\frac{1}{2}}$
5942.72	+ 1 − 3	150	III	16822.66	$a^6D^\circ_{2\frac{1}{2}}-z^6F_{2\frac{1}{2}}$ $b^8D^\circ_{2\frac{1}{2}}-y^8D_{1\frac{1}{2}}$
5937.77	0 +13	15	IV	16836.68	$b^8D^\circ_{3\frac{1}{2}}-v^8P_{4\frac{1}{2}}$ $(b^8D^\circ_{2\frac{1}{2}}-u^8P_{3\frac{1}{2}})$
5927.03	+ 1	1	IV	16867.19	$a^6D^\circ_{1\frac{1}{2}}-v^8P_{2\frac{1}{2}}$
5926.52	0	300	III	16868.65	$a^8D^\circ_{5\frac{1}{2}}-y^{10}P_{5\frac{1}{2}}$
5925.30	− 1	40	III	16872.12	$b^8D^\circ_{3\frac{1}{2}}-y^8D_{2\frac{1}{2}}$
5924.91	0	15	III	16873.23	$b^8D^\circ_{4\frac{1}{2}}-u^8P_{4\frac{1}{2}}$
5915.74	0	800	III	16899.38	$z^{10}P_{5\frac{1}{2}}-e^{10}P^\circ_{5\frac{1}{2}}$
5914.66	+ 1	20	III	16902.47	$a^6D^\circ_{\frac{1}{2}}-z^6F_{1\frac{1}{2}}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
5909.94	−1	60	III A	16915.97	$a^8D^\circ_{4\frac{1}{2}}-y^{10}P_{4\frac{1}{2}}$
5909.41	+3	10	IV A	16917.49	$b^8D^\circ_{4\frac{1}{2}}-v^8P_{4\frac{1}{2}}$
5908.76	+1	8	IV A	16919.35	$a^6D^\circ_{2\frac{1}{2}}-v^8P_{3\frac{1}{2}}$
5902.77	+1	125	III	16936.52	$b^8D^\circ_{2\frac{1}{2}}-y^8D_{3\frac{1}{2}}$
5895.31	0	40	III	16957.95	$a^6D^\circ_{1\frac{1}{2}}-z^6F_{1\frac{1}{2}}$
5891.30	+3	10	IV	16969.49	$b^8D^\circ_{4\frac{1}{2}}-u^8P_{3\frac{1}{2}}$
5888.72	0	1	IV A	16976.93	$a^8D^\circ_{3\frac{1}{2}}-y^{10}P_{3\frac{1}{2}}$
5885.15	0	10	IV	16987.22	$a^6D^\circ_{\frac{1}{2}}-z^6F_{\frac{1}{2}}$
5884.83	+1	15	IV	16988.15	$b^8D^\circ_{3\frac{1}{2}}-y^8D_{3\frac{1}{2}}$
5880.42	−1	1§	IV A	17000.89	$a^8D^\circ_{4\frac{1}{2}}-118_{3\frac{1}{2}}$
5874.87	0	1	IV	17016.95	$a^6D^\circ_{3\frac{1}{2}}-v^8P_{3\frac{1}{2}}$
5866.67	+1	40	III A	17040.73	$a^{10}D^\circ_{6\frac{1}{2}}-111_{5\frac{1}{2}}$
5864.77	+6	40	III	17046.25	$(a^6D^\circ_{3\frac{1}{2}}-v^8P_{2\frac{1}{2}})$
5860.97	0	80	III	17057.31	$b^8D^\circ_{5\frac{1}{2}}-u^8P_{4\frac{1}{2}}$
5856.95	+2	20	III	17069.01	$b^8D^\circ_{4\frac{1}{2}}-y^8D_{3\frac{1}{2}}$
5854.13	+1	1	IV	17077.23	$a^6D^\circ_{3\frac{1}{2}}-u^8P_{2\frac{1}{2}}$
5852.42	+2	3	IV A	17082.22	$z^6P_{3\frac{1}{2}}-e^{10}D^\circ_{2\frac{1}{2}}$
5846.37	−2	1	IV A	17099.90	$z^6P_{3\frac{1}{2}}-e^{10}D^\circ_{3\frac{1}{2}}$
5845.77	−1	50	III	17101.66	$b^8D^\circ_{5\frac{1}{2}}-v^8P_{4\frac{1}{2}}$
5843.55	+2	12	IV	17108.15	$a^6D^\circ_{4\frac{1}{2}}-v^8P_{3\frac{1}{2}}$
5838.03	−1	15	III	17124.33	$a^6D^\circ_{2\frac{1}{2}}-y^8D_{1\frac{1}{2}}$
5832.68	−2	1	IV	17140.04	$a^6D^\circ_{3\frac{1}{2}}-u^8P_{4\frac{1}{2}}$
5830.98	0	5000	II	17145.03	$a^{10}D^\circ_{6\frac{1}{2}}-z^{10}F_{7\frac{1}{2}}$
5829.49	+1 0	50	III A	17149.42	$z^{10}P_{3\frac{1}{2}}-f^8S^\circ_{3\frac{1}{2}}$ $(a^8D^\circ_{2\frac{1}{2}}-y^{10}P_{3\frac{1}{2}})$
5820.80	−1	25§	IV	17175.02	$a^8D^\circ_{3\frac{1}{2}}-y^{10}P_{4\frac{1}{2}}$
5820.03	+4	50	III	17177.29	$b^8D^\circ_{3\frac{1}{2}}-y^8D_{4\frac{1}{2}}$
5817.65	0	1	IV	17184.32	$a^6D^\circ_{3\frac{1}{2}}-v^8P_{4\frac{1}{2}}$
5805.68	0	20	III	17219.75	$a^6D^\circ_{3\frac{1}{2}}-y^8D_{2\frac{1}{2}}$
5800.27	0 −1 +17	800	II	17235.81	$a^{10}D^\circ_{6\frac{1}{2}}-z^8D_{5\frac{1}{2}}$ $(b^8D^\circ_{4\frac{1}{2}}-y^8D_{3\frac{1}{2}})$ $(a^6D^\circ_{3\frac{1}{2}}-u^8P_{3\frac{1}{2}})$
5792.72	+1	50	III	17258.27	$b^8D^\circ_{4\frac{1}{2}}-y^8D_{4\frac{1}{2}}$
5792.08	+3	1	IV	17260.18	$z^6P_{2\frac{1}{2}}-e^8D^\circ_{2\frac{1}{2}}$
5789.41	0	1	IV A	17268.14	$a^8D^\circ_{4\frac{1}{2}}-y^{10}P_{5\frac{1}{2}}$
5783.69	+1	1200	II	17285.22	$z^{10}P_{4\frac{1}{2}}-e^{10}P^\circ_{3\frac{1}{2}}$
5778.90	0	1	IV	17299.54	$b^8D^\circ_{1\frac{1}{2}}-133_{2\frac{1}{2}}$
5769.88	−1	1	IV A	17326.59	$z^6P_{2\frac{1}{2}}-e^8D^\circ_{3\frac{1}{2}}$
5769.56	+1	3	IV	17327.55	$a^6D^\circ_{4\frac{1}{2}}-u^8P_{3\frac{1}{2}}$
5766.80	0	3	IV	17335.84	$a^6D^\circ_{3\frac{1}{2}}-y^8D_{3\frac{1}{2}}$
5765.20	0	2500	I	17340.65	$a^8S^\circ_{3\frac{1}{2}}-z^6P_{3\frac{1}{2}}$
5758.03	−1	1−	IV A	17362.25	$a^{10}D^\circ_{5\frac{1}{2}}-111_{5\frac{1}{2}}$
5744.36	0	12	IV	17403.56	$b^8D^\circ_{4\frac{1}{2}}-132_{5\frac{1}{2}}$
5740.9	−6	1	V	17414.1	$b^8D^\circ_{3\frac{1}{2}}-134_{4\frac{1}{2}}$
5739.000	+1	400	III	17419.82	$b^8D^\circ_{5\frac{1}{2}}-y^8D_{5\frac{1}{2}}$
5736.61	+1	12	IV	17427.07	$a^6D^\circ_{4\frac{1}{2}}-y^8D_{3\frac{1}{2}}$
5731.56	−2	6	IV	17442.43	$b^8D^\circ_{5\frac{1}{2}}-y^8D_{4\frac{1}{2}}$
5730.87	0	500	II	17444.53	$a^{10}D^\circ_{5\frac{1}{2}}-z^8D_{4\frac{1}{2}}$
5728.20	0	12	IV	17452.66	$z^6P_{1\frac{1}{2}}-f^8D^\circ_{2\frac{1}{2}}$
5721.87	+4	1	IV	17471.97	$b^8D^\circ_{1\frac{1}{2}}-135_{2\frac{1}{2}}$
5718.81	−1	8	IV	17481.32	$y^8P_{4\frac{1}{2}}-i^8S^\circ_{3\frac{1}{2}}$
5714.39	−1	1	IV	17494.84	$b^8D^\circ_{4\frac{1}{2}}-134_{4\frac{1}{2}}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
5707.89	−4	4	IV A	17514.76	$z^8P_{4½} - 3^\circ_{3½, 4½}$
5696.3	−11	1−	IV	17550.4	$a^6D^\circ_{1½} - 133_{2½}$?
5694.05	−2	1	IV A	17557.33	$a^{10}D^\circ_{5½} - z^8D_{5½}$
5692.68	+1	1	IV A	17561.56	$a^{10}D^\circ_{4½} - 109_{3½}$
5688.38	−1	20	III A	17574.83	$z^6P_{2½} - f^8D^\circ_{1½}$
5684.240	0	80	III	17587.63	$b^8D^\circ_{5½} - 132_{5½}$
5682.25	0	1−	IV A	17593.79	$a^6D^\circ_{4½} - y^8D_{5½}$
5681.099	+1	100	III	17597.36	$a^{10}D^\circ_{4½} - 111_{5½}$
5674.98	0	10	IV	17616.33	$a^6D^\circ_{4½} - y^8D_{4½}$
5673.85	0	600	II	17619.84	$a^{10}D^\circ_{4½} - z^8D_{3½}$
5671.53	0	1−	IV A	17627.05	$z^6P_{3½} - e^8D^\circ_{2½}$
5670.1	+6	1	IV	17631.5	$a^6D^\circ_{2½} - 133_{2½}$
5668.23	+3	6	IV	17637.31	$y^8P_{3½} - i^8S^\circ_{3½}$
5665.35	+1	8	III A	17646.28	$z^{10}P_{4½} - e^{10}P^\circ_{4½}$
5657.57	−1	1	IV	17670.54	$z^6P_{2½} - f^8D^\circ_{3½}$
5654.65	+1	10	III A	17679.67	$a^{10}D^\circ_{4½} - z^8D_{4½}$
5651.11	0	30	III	17690.74	$z^6P_{2½} - f^8D^\circ_{2½}$
5650.28	0	8	IV	17693.34	$z^6P_{3½} - e^8D^\circ_{3½}$
5649.88	−1	8	IV A	17694.59	$z^8P_{4½} - 4^\circ$
5645.795	+1	1200	I	17707.40	$a^8S^\circ_{3½} - z^6P_{2½}$
5640.22	+1	5	IV	17724.90	$y^8P_{4½} - h^8D^\circ_{5½}$
5638.41	−11	2	IV	17730.59	$y^8P_{4½} - h^8D^\circ_{4½}$?
5635.16	+3	2	IV A	17740.81	$b^8D^\circ_{2½} - 138_{3½}$
5632.54	0	600	II	17749.06	$a^{10}D^\circ_{3½} - z^8D_{2½}$
5627.07	−1	10	IV	17766.32	$z^6P_{3½} - e^8D^\circ_{4½}$
5622.44	0	600	II	17780.95	$z^{10}P_{3½} - e^{10}P^\circ_{3½}$
5619.50	+1	1	IV	17790.25	$b^8D^\circ_{3½} - 137_{2½, 3½}$
5618.81	{ 0 +17 }	150	III	17792.44	{ $a^{10}D^\circ_{4½} - z^8D_{5½}$ $(a^{10}D^\circ_{3½} - z^8D_{3½})$
5617.05	−1	4	IV	17798.01	$y^8P_{2½} - i^8S^\circ_{3½}$
5607.38	0	.8	IV A	17828.70	$z^8P_{4½} - e^{10}D^\circ_{3½}$
5605.86	0	60	III	17833.54	$z^6P_{3½} - e^8P^\circ_{2½}$?
5599.80	0	60	III	17852.84	$a^{10}D^\circ_{3½} - z^8D_{4½}$
5599.11	−1	15	IV	17855.04	$z^8P_{4½} - e^{10}D^\circ_{4½}$
5593.10	+1	5	IV A	17874.22	$a^{10}D^\circ_{2½} - z^8D_{2½}$
5592.25	+2	40	III	17876.94	$a^{10}D^\circ_{2½} - z^8D_{1½}$
5591.60	−2	4	IV	17879.02	$b^8D^\circ_{3½} - 141_{4½}$
5589.28	−4	2	IV A	17886.44	$y^8P_{3½} - h^8D^\circ_{4½}$
5587.73	0	1	IV A	17891.40	$y^8P_{3½} - h^8D^\circ_{3½}$
5586.83	{ +2 −16 }	600	III	17894.28	{ $a^{10}D^\circ_{4½} - 113_{3½}$ $(z^8P_{4½} - e^{10}D^\circ_{5½})$
5586.24	0	800	II	17896.17	$a^{10}D^\circ_{3½} - z^{10}D_{2½}$
5580.03	−1	800	II	17916.09	$a^{10}D^\circ_{4½} - z^{10}D_{3½}$
5579.63	{ 0 +25 }	600	III	17917.37	{ $z^{10}P_{4½} - e^{10}P^\circ_{5½}$ $(a^{10}D^\circ_{2½} - z^8D_{3½})$
5577.14	−1	1500	II	17925.37	$a^{10}D^\circ_{5½} - z^{10}D_{4½}$
5572.65	−1	4	IV A	17939.82	$z^6P_{3½} - e^8P^\circ_{4½}$
5570.33	−1	1000	II	17947.29	$a^{10}D^\circ_{6½} - z^{10}D_{5½}$
5566.41	−2	4	IV	17959.92	$b^8D^\circ_{4½} - 141_{4½}$
5547.44	0	1200	II	18021.34	$a^{1c}D^\circ_{2½} - z^{10}D_{2½}$
5542.54	+1	100	III	18037.27	$z^6P_{3½} - f^8D^\circ_{3½}$
5541.60	+1	20	IV	18040.33	$a^6D^\circ_{2½} - 137_{2½, 3½}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
5538.02	0	2	IV	18051.99	$y^8P_{2\frac{1}{2}}-h^8D^{\circ}_{3\frac{1}{2}}$
5536.83	−4	2	IV	18055.87	$y^8P_{2\frac{1}{2}}-h^8D^{\circ}_{2\frac{1}{2}}$
5536.10	+4	4	III A	18058.26	$a^{10}D^{\circ}_{3\frac{1}{2}}-112_{2\frac{1}{2}}$
5533.25	−2	80	III	18067.55	$a^{10}D^{\circ}_{3\frac{1}{2}}-113_{3\frac{1}{2}}$
5526.627	0	125	III	18089.21	$a^{10}D^{\circ}_{3\frac{1}{2}}-z^{10}D_{3\frac{1}{2}}$
5524.44	+6	1	IV A	18096.37	$a^{10}D^{\circ}_{5\frac{1}{2}}-114_{4\frac{1}{2}}$?
5519.62	+1	3	IV A	18112.17	$a^6D^{\circ}_{2\frac{1}{2}}-139_{2\frac{1}{2},\,3\frac{1}{2}}$
5511.77	−2	4	IV	18137.97	$a^6D^{\circ}_{3\frac{1}{2}}-137_{2\frac{1}{2},\,3\frac{1}{2}}$
5511.09	−1	20	III	18140.20	$a^6D^{\circ}_{3\frac{1}{2}}-138_{3\frac{1}{2}}$
5510.52	−2	600	II	18142.08	$z^{10}P_{3\frac{1}{2}}-e^{10}P^{\circ}_{4\frac{1}{2}}$
5504.93	+1	3	III A	18160.50	$a^{10}D^{\circ}_{4\frac{1}{2}}-z^{10}D_{4\frac{1}{2}}$
5502.30	−1	2	IV	18169.18	$b^8D^{\circ}_{4\frac{1}{2}}-142_{5\frac{1}{2}}$
5500.83	+2	100	III	18174.04	$z^8P_{3\frac{1}{2}}-3^{\circ}_{3\frac{1}{2},\,4\frac{1}{2}}$
5500.48	0	20	III A	18175.20	$z^8P_{2\frac{1}{2}}-1^{\circ}_{2\frac{1}{2}}$
5497.95	0	15	III A	18183.56	$a^{10}D^{\circ}_{2\frac{1}{2}}-112_{2\frac{1}{2}}$
5495.80	−1	10	III A	18190.67	$z^8P_{2\frac{1}{2}}-2^{\circ}_{1\frac{1}{2},\,2\frac{1}{2}}$
5495.200	0	250	II	18192.66	$a^{10}D^{\circ}_{2\frac{1}{2}}-113_{3\frac{1}{2}}$
5490.03	−1	3	IV A	18209.79	$a^6D^{\circ}_{3\frac{1}{2}}-139_{2\frac{1}{2},\,3\frac{1}{2}}$
5488.653	+1	800	II	18214.36	$a^{10}D^{\circ}_{2\frac{1}{2}}-z^{10}D_{3\frac{1}{2}}$
5485.50	+1	15	IV	18224.83	$a^6D^{\circ}_{3\frac{1}{2}}-140_{3\frac{1}{2}}$
5484.40	0	8	III A	18228.48	$a^8D^{\circ}_{4\frac{1}{2}}-120_{3\frac{1}{2}}$
5483.52	+1	1	IV	18231.41	$a^6D^{\circ}_{4\frac{1}{2}}-138_{3\frac{1}{2}}$
5481.78	0	3	IV A	18237.20	$a^8D^{\circ}_{5\frac{1}{2}}-x^{10}P_{5\frac{1}{2}}$
5472.324	0	600	II	18268.71	$a^{10}D^{\circ}_{5\frac{1}{2}}-z^{10}D_{5\frac{1}{2}}$
5467.05	+1	20	III A	18286.33	$a^8D^{\circ}_{5\frac{1}{2}}-w^8P_{4\frac{1}{2}}$
5458.15	−1	1	IV	18316.15	$a^6D^{\circ}_{4\frac{1}{2}}-140_{3\frac{1}{2}}$
5457.62	−1	40	III	18317.93	$a^6D^{\circ}_{4\frac{1}{2}}-141_{4\frac{1}{2}}$
5452.94	0	1200	II	18333.65	$a^{10}D^{\circ}_{3\frac{1}{2}}-z^{10}D_{4\frac{1}{2}}$
5451.51	−1 −15	1500	II	18338.46	$a^{10}D^{\circ}_{6\frac{1}{2}}-z^{10}D_{6\frac{1}{2}}$ $(z^6P_{2\frac{1}{2}}-g^8D^{\circ}_{1\frac{1}{2}})$
5447.13	+1	8	IV	18353.20	$b^8D^{\circ}_{5\frac{1}{2}}-142_{5\frac{1}{2}}$
5443.564	0 −1	250	III	18365.23	$a^8D^{\circ}_{3\frac{1}{2}}-x^8P_{2\frac{1}{2}}$ $(z^6P_{2\frac{1}{2}}-g^8D^{\circ}_{2\frac{1}{2}})$
5436.22	+5	2	IV	18390.04	$z^6P_{2\frac{1}{2}}-g^8D^{\circ}_{3\frac{1}{2}}$
5426.943	+1	400	III	18421.47	$a^8D^{\circ}_{4\frac{1}{2}}-x^8P_{3\frac{1}{2}}$
5426.8	+6	100?	III	18422.0	$z^8P_{4\frac{1}{2}}-e^8D^{\circ}_{3\frac{1}{2}}$
5421.074	+1	200	III	18441.42	$z^8P_{4\frac{1}{2}}-e^8P^{\circ}_{3\frac{1}{2}}$
5419.05	0	1	IV A	18448.30	$a^6D^{\circ}_{3\frac{1}{2}}-143_{3\frac{1}{2}}$
5416.27	+2	3	IV	18457.77	$a^8D^{\circ}_{3\frac{1}{2}}-119_{3\frac{1}{2}}$
5413.79†	+2	15	III A	18466.23	$a^8D^{\circ}_{5\frac{1}{2}}-x^{10}P_{4\frac{1}{2}}$
5412.52	+2	8	III A	18470.56	$z^8P_{3\frac{1}{2}}-e^{10}D^{\circ}_{2\frac{1}{2}}$
5411.864	+1	100	III	18472.80	$z^6P_{3\frac{1}{2}}-f^8D^{\circ}_{4\frac{1}{2}}$
5407.42†	+6	8	III A	18487.98	$z^8P_{3\frac{1}{2}}-e^{10}D^{\circ}_{3\frac{1}{2}}$
5405.33	+1	125	III	18495.13	$z^8P_{4\frac{1}{2}}-e^8D^{\circ}_{4\frac{1}{2}}$
5402.77	0	1200	II	18503.89	$a^{10}D^{\circ}_{4\frac{1}{2}}-z^{10}D_{5\frac{1}{2}}$
5399.68	0	3	IV A	18514.48	$z^8P_{3\frac{1}{2}}-e^{10}D^{\circ}_{4\frac{1}{2}}$
5396.01	+2 +3	3	IV	18527.07	$b^8D^{\circ}_{2\frac{1}{2}}-148_{3\frac{1}{2}}$ $a^6D^{\circ}_{4\frac{1}{2}}-142_{5\frac{1}{2}}$
5393.47	+1	2	IV	18535.80	$b^8D^{\circ}_{4\frac{1}{2}}-z^6D_{4\frac{1}{2}}$
5392.94	+3	250	III	18537.62	$a^8D^{\circ}_{2\frac{1}{2}}-x^8P_{2\frac{1}{2}}$
5391.00	−2	3	IV	18544.29	$a^6D^{\circ}_{3\frac{1}{2}}-145_{4\frac{1}{2}}$
5376.939	0	300	III	18592.78	$z^8P_{4\frac{1}{2}}-e^8D^{\circ}_{5\frac{1}{2}}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
5371.80	0	2	IV A	18610.57	$a^{6}D^{\circ}_{2\frac{1}{2}}-146_{3\frac{1}{2}}$
5364.62	0	8	IV	18635.48	$a^{6}D^{\circ}_{4\frac{1}{2}}-145_{4\frac{1}{2}}$
5364.2	− 6	1−	IV A	18636.9	$a^{8}D^{\circ}_{4\frac{1}{2}}-x^{10}P_{5\frac{1}{2}}$?
5361.61	0	500	III	18645.94	$a^{8}D^{\circ}_{5\frac{1}{2}}-x^{8}P_{4\frac{1}{2}}$
5360.83	+ 3	250	III	18648.66	$a^{8}D^{\circ}_{1\frac{1}{2}}-x^{8}P_{2\frac{1}{2}}$
5357.8	− 5	[50?]	III A	18659.2	$a^{8}D^{\circ}_{5\frac{1}{2}}-122_{4\frac{1}{2}}$
5357.61	+ 1	1200	II	18659.86	$a^{10}D^{\circ}_{5\frac{1}{2}}-z^{10}D_{6\frac{1}{2}}$
5356.73	0	60	III	18662.93	$z^{6}P_{1\frac{1}{2}}-e^{6}D^{\circ}_{2\frac{1}{2}}$
5355.10	+ 1	300	III	18668.61	$z^{8}P_{4\frac{1}{2}}-e^{8}P^{\circ}_{4\frac{1}{2}}$
5353.84	+ 1	1	IV A	18673.00	$a^{10}D^{\circ}_{5\frac{1}{2}}-115_{4\frac{1}{2}}$
5352.84	+ 1	100	III	18676.49	$z^{6}P_{1\frac{1}{2}}-e^{6}D^{\circ}_{1\frac{1}{2}}$
5351.69	+ 1	300	III	18680.50	$a^{8}D^{\circ}_{3\frac{1}{2}}-x^{8}P_{3\frac{1}{2}}$
5350.41	0	80	III	18684.97	$z^{6}P_{1\frac{1}{2}}-e^{6}D^{\circ}_{\frac{1}{2}}$
5350.14	− 2	5?	IV A	18685.92	$a^{8}D^{\circ}_{4\frac{1}{2}}-w^{8}P_{4\frac{1}{2}}$
5348.18	+ 3	1	IV	18692.76	$y^{8}P_{3\frac{1}{2}}-f^{10}P^{\circ}_{3\frac{1}{2}}$
5346.22	− 1	1	IV	18699.62	$b^{8}D^{\circ}_{3\frac{1}{2}}-z^{6}D_{3\frac{1}{2}}$
5343.79	0	3	IV	18708.12	$a^{6}D^{\circ}_{3\frac{1}{2}}-146_{3\frac{1}{2}}$
5338.30	{+ 1 − 6}	8	III A	18727.36	{ $a^{10}D^{\circ}_{5\frac{1}{2}}-z^{8}F_{4\frac{1}{2}}$ ($a^{10}D^{\circ}_{3\frac{1}{2}}-z^{8}F_{2\frac{1}{2}}$)
5336.97	− 2	2	IV A	18732.03	$z^{6}P_{3\frac{1}{2}}-g^{8}D^{\circ}_{2\frac{1}{2}}$
5335.64	0	1	IV	18736.70	$b^{8}D^{\circ}_{1\frac{1}{2}}-z^{6}D_{1\frac{1}{2}}$
5333.28	+ 3	1	V	18744.99	$b^{8}D^{\circ}_{3\frac{1}{2}}-z^{6}D_{2\frac{1}{2}}$
5329.89	+ 2	1	IV A	18756.91	$z^{6}P_{3\frac{1}{2}}-g^{8}D^{\circ}_{3\frac{1}{2}}$
5327.3	+ 4	1	IV ?	18766.0	$z^{8}P_{4\frac{1}{2}}-f^{8}D^{\circ}_{3\frac{1}{2}}$
5326.2	− 4	1	V	18769.9	$b^{8}D^{\circ}_{1\frac{1}{2}}-z^{6}D_{\frac{1}{2}}$
5323.02	0	6	III A	18781.12	$a^{10}D^{\circ}_{4\frac{1}{2}}-z^{8}F_{3\frac{1}{2}}$
5316.94	0	20	IV	18802.59	$a^{6}D^{\circ}_{3\frac{1}{2}}-z^{6}D_{4\frac{1}{2}}$
5312.22	+ 1	5	III A	18819.30	$a^{10}D^{\circ}_{6\frac{1}{2}}-117_{5\frac{1}{2},\,6\frac{1}{2}}$
5310.01	+ 1	3	III A	18827.13	$a^{8}D^{\circ}_{1\frac{1}{2}}-121_{2\frac{1}{2}}$
5305.50	0	5	IV	18843.14	$a^{6}D^{\circ}_{2\frac{1}{2}}-150_{3\frac{1}{2}}$
5303.85	− 1	300	III	18849.00	$z^{8}P_{4\frac{1}{2}}-f^{8}D^{\circ}_{5\frac{1}{2}}$
5302.72	{ 0 +12}	80	III	18853.01	{ $a^{8}D^{\circ}_{2\frac{1}{2}}-x^{8}P_{3\frac{1}{2}}$ ($y^{8}P_{2\frac{1}{2}}-f^{10}P^{\circ}_{3\frac{1}{2}}$)
5299.63	− 1	1	IV A	18864.01	$a^{10}D^{\circ}_{2\frac{1}{2}}-z^{8}F_{1\frac{1}{2}}$
5299.14	+ 1	1	IV A	18865.75	$a^{8}D^{\circ}_{4\frac{1}{2}}-x^{10}P_{4\frac{1}{2}}$
5298.10	+ 2	40	III A	18869.46	$a^{10}D^{\circ}_{5\frac{1}{2}}-116_{4\frac{1}{2},\,5\frac{1}{2}}$
5295.58	− 2	2	{IV V E}	18878.43	$z^{6}P_{3\frac{1}{2}}-g^{8}D^{\circ}_{4\frac{1}{2}}$
5294.7	− 4 }	700	{IV	18881.6	$a^{8}D^{\circ}_{4\frac{1}{2}}-w^{8}P_{3\frac{1}{2}}$
5294.64	+ 2 }		III	18881.78	$z^{6}P_{2\frac{1}{2}}-e^{6}D^{\circ}_{3\frac{1}{2}}$
5293.68	0	200	III	18885.21	$z^{10}P_{5\frac{1}{2}}-e^{10}D^{\circ}_{4\frac{1}{2}}$
5292.54	− 1	1	IV A	18889.28	$y^{8}P_{4\frac{1}{2}}-7^{\circ}_{3\frac{1}{2}}$
5291.26	0	200	III	18893.85	$a^{6}D^{\circ}_{4\frac{1}{2}}-z^{6}D_{4\frac{1}{2}}$
5289.25	− 1	300	III	18901.03	$z^{6}P_{2\frac{1}{2}}-e^{6}D^{\circ}_{2\frac{1}{2}}$
5287.25	+ 1	150	III	18908.18	$a^{10}D^{\circ}_{4\frac{1}{2}}-115_{4\frac{1}{2}}$
5285.73	− 1	60	III	18913.61	$a^{6}D^{\circ}_{1\frac{1}{2}}-z^{6}D_{2\frac{1}{2}}$
5285.47	+ 1	40	III	18914.54	$z^{6}P_{2\frac{1}{2}}-e^{6}D^{\circ}_{1\frac{1}{2}}$
5282.82	− 2	1000	III	18924.03	$z^{10}P_{5\frac{1}{2}}-e^{10}D^{\circ}_{5\frac{1}{2}}$
5280.65	− 1	25	III	18931.81	$a^{6}D^{\circ}_{\frac{1}{2}}-z^{6}D_{1\frac{1}{2}}$
5278.17	0	20	III	18940.70	$a^{6}D^{\circ}_{3\frac{1}{2}}-150_{3\frac{1}{2}}$
5277.05	+ 4	2	IV A	18944.72	$a^{8}D^{\circ}_{3\frac{1}{2}}-w^{8}P_{4\frac{1}{2}}$
5275.66	0	50	III	18949.71	$a^{6}D^{\circ}_{2\frac{1}{2}}-z^{6}D_{3\frac{1}{2}}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
5275.05	−1 +11	15	III	18951.91	$z^6P_{2\frac{1}{2}}-g^8S^{\circ}_{3\frac{1}{2}}$ $(a^{10}D^{\circ}_{2\frac{1}{2}}-z^8F_{2\frac{1}{2}})$
5274.39	0	8	III A	18954.28	$a^{10}D^{\circ}_{3\frac{1}{2}}-z^8F_{3\frac{1}{2}}$
5272.48	+1	500	II	18961.14	$a^{10}D^{\circ}_{5\frac{1}{2}}-z^8F_{5\frac{1}{2}}$
5271.96	0 −13	3000	III	18963.01	$z^{10}P_{5\frac{1}{2}}-e^{10}D^{\circ}_{6\frac{1}{2}}$ $(a^{10}D^{\circ}_{4\frac{1}{2}}-z^8F_{4\frac{1}{2}})$
5271.47	+1	5	IV	18964.78	$a^6D^{\circ}_{\frac{1}{2}}-z^6D_{\frac{1}{2}}$
5268.64	−2	2	IV	18974.96	$b^8D^{\circ}_{2\frac{1}{2}}-155_{3\frac{1}{2}}$
5266.40	+1	1200	II	18983.03	$a^{10}D^{\circ}_{6\frac{1}{2}}-z^8F_{6\frac{1}{2}}$
5265.22	−2	2	IV	18987.29	$a^6D^{\circ}_{1\frac{1}{2}}-z^6D_{1\frac{1}{2}}$
5263.03	+1	50	III	18995.19	$a^6D^{\circ}_{2\frac{1}{2}}-z^6D_{2\frac{1}{2}}$
5260.92	0	2	IV	19002.81	$y^8P_{4\frac{1}{2}}-8^{\circ}_{4\frac{1}{2},\,5\frac{1}{2}}$
5257.44	0	1	IV	19015.39	$z^8P_{3\frac{1}{2}}-e^8D^{\circ}_{2\frac{1}{2}}$
5256.79	−1	1−	V	19017.74	$a^6D^{\circ}_{4\frac{1}{2}}-148_{3\frac{1}{2}}$
5256.08	−1	30	III	19020.31	$a^6D^{\circ}_{1\frac{1}{2}}-z^6D_{\frac{1}{2}}$
5255.32	0	1	IV A	19023.06	$z^8P_{2\frac{1}{2}}-e^8D^{\circ}_{1\frac{1}{2}}$
5253.38	+2	2	V	19030.08	$b^8D^{\circ}_{4\frac{1}{2}}-153_{4\frac{1}{2}}$
5252.86	0	2	IV A	19031.97	$a^6D^{\circ}_{4\frac{1}{2}}-150_{3\frac{1}{2}}$
5249.17	+2	150	III	19045.34	$a^8D^{\circ}_{4\frac{1}{2}}-x^8P_{4\frac{1}{2}}$
5248.65	+1	150	III	19047.23	$a^6D^{\circ}_{3\frac{1}{2}}-z^6D_{3\frac{1}{2}}$
5245.55	+1	6	III A	19058.49	$a^8D^{\circ}_{4\frac{1}{2}}-122_{4\frac{1}{2}}$
5242.71	+2	40	III	19068.81	$a^6D^{\circ}_{2\frac{1}{2}}-z^6D_{1\frac{1}{2}}$
5240.43	+1	1	IV A	19077.11	$z^8P_{2\frac{1}{2}}-e^8D^{\circ}_{2\frac{1}{2}}$
5239.82	+3	2	III A	19079.33	$a^{10}D^{\circ}_{2\frac{1}{2}}-z^8F_{3\frac{1}{2}}$
5239.24	−2 +6	150	III	19081.44	$a^{10}D^{\circ}_{3\frac{1}{2}}-115_{4\frac{1}{2}}$ $z^8P_{3\frac{1}{2}}-e^8D^{\circ}_{3\frac{1}{2}}$
5236.13	−1 +2	30	III	19092.78	$a^6D^{\circ}_{3\frac{1}{2}}-z^6D_{2\frac{1}{2}}$ $(y^8P_{4\frac{1}{2}}-f^{10}P^{\circ}_{4\frac{1}{2}})$
5233.90	0	50	III	19100.91	$z^8P_{3\frac{1}{2}}-e^8P^{\circ}_{3\frac{1}{2}}$
5232.89	+3	1	IV A	19104.59	$a^{10}D^{\circ}_{4\frac{1}{2}}-116_{4\frac{1}{2},\,5\frac{1}{2}}$
5227.3	−5	1	IV	19125.0	$a^8D^{\circ}_{3\frac{1}{2}}-x^{10}P_{4\frac{1}{2}}$
5224.68	−1	3	III A	19134.62	$a^8D^{\circ}_{3\frac{1}{2}}-w^8P_{2\frac{1}{2}}$
5224.41	+2	1	IV A	19135.60	$a^{10}D^{\circ}_{3\frac{1}{2}}-z^8F_{4\frac{1}{2}}$
5223.49	+1 −13	1000	II	19138.97	$a^{10}D^{\circ}_{5\frac{1}{2}}-y^{10}P_{4\frac{1}{2}}$ $(a^6D^{\circ}_{4\frac{1}{2}}-z^6D_{3\frac{1}{2}})$
5223.09	+1	20	III A	19140.44	$a^8D^{\circ}_{3\frac{1}{2}}-w^8P_{3\frac{1}{2}}$
5222.28	+1	2	IV A	19143.41	$z^8P_{2\frac{1}{2}}-e^8D^{\circ}_{3\frac{1}{2}}$
5219.42	−2	4	IV	19153.90	$b^8D^{\circ}_{2\frac{1}{2}}-y^6P_{3\frac{1}{2}}$
5219.22	0	10	III	19154.63	$z^8P_{3\frac{1}{2}}-e^8D^{\circ}_{4\frac{1}{2}}$
5217.01	−1	200	III	19162.75	$z^8P_{2\frac{1}{2}}-e^8P^{\circ}_{3\frac{1}{2}}$
5215.10	−1	2000	II	19169.76	$a^{10}D^{\circ}_{6\frac{1}{2}}-y^{10}P_{5\frac{1}{2}}$
5213.36	−1	300	III	19176.16	$a^{10}D^{\circ}_{4\frac{1}{2}}-y^{10}P_{3\frac{1}{2}}$
5207.88	0	20	III A	19196.34	$a^{10}D^{\circ}_{4\frac{1}{2}}-z^8F_{5\frac{1}{2}}$
5206.44	+1	60	III	19201.65	$z^8P_{4\frac{1}{2}}-f^8D^{\circ}_{4\frac{1}{2}}$
5202.58	+1	1	IV	19215.90	$a^6D^{\circ}_{3\frac{1}{2}}-y^{10}F_{4\frac{1}{2}}$
5200.96	0	400	III	19221.88	$z^8P_{3\frac{1}{2}}-e^8P^{\circ}_{2\frac{1}{2}}$
5199.85	0	800	III	19225.98	$z^6P_{3\frac{1}{2}}-e^6D^{\circ}_{4\frac{1}{2}}$
5199.27	−1	4	IV	19228.13	$b^8D^{\circ}_{1\frac{1}{2}}-y^6P_{2\frac{1}{2}}$
5198.22	−2	2	IV	19232.01	$y^8P_{4\frac{1}{2}}-10^{\circ}_{3\frac{1}{2},\,4\frac{1}{2}}$
5193.74	+1 +9	150	III	19248.60	$z^6P_{3\frac{1}{2}}-e^6D^{\circ}_{3\frac{1}{2}}$ $(y^8P_{3\frac{1}{2}}-f^{10}P^{\circ}_{4\frac{1}{2}})$
5189.88	−1	10	IV	19262.92	$b^8D^{\circ}_{3\frac{1}{2}}-157_{4\frac{1}{2}}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
5188.59	+2	10	III	19267.71	$z^6P_{3½}-e^6D^\circ_{2½}$
5184.30	+1	2	IV A	19283.65	$z^8P_{2½}-e^8P^\circ_{2½}$
5183.56	−1	4	IV	19286.40	$b^8D^\circ_{4½}-y^6P_{3½}$
5182.14	−3	1	IV	19291.69	$b^8D^\circ_{1½}-y^8F_{2½}$
5181.03	−1	12	IV	19295.82	$b^8D^\circ_{2½}-y^8F_{3½}$
5178.69	{−2 0}	50	III A	19304.54	{$a^8D^\circ_{3½}-x^8P_{4½}$ $a^{10}D^\circ_{5½}-z^8F_{6½}$}
5178.01	−1	25	III A	19307.08	$a^8D^\circ_{2½}-w^8P_{2½}$
5177.17	−1	2	IV	19310.21	$b^8D^\circ_{3½}-y^6P_{2½}$
5176.42	−1	6	III A	19313.01	$a^8D^\circ_{2½}-w^8P_{3½}$
5175.20	0	1	IV A	19317.56	$a^8D^\circ_{3½}-122_{4½}$
5174.91	+1	30	III	19318.64	$z^6P_{3½}-g^8S^\circ_{3½}$
5174.02	+1	12	IV	19321.96	$b^8D^\circ_{2½}-y^8F_{2½}$
5172.39	+2	1	IV A	19328.05	$z^8P_{3½}-e^8P^\circ_{4½}$
5170.50	0	20	IV	19335.12	$b^8D^\circ_{1½}-y^8F_{1½}$
5169.75	−1	20	IV	19337.93	$b^8D^\circ_{1½}-y^8F_{½}$
5169.32	−1	40	III	19339.53	$b^8D^\circ_{3½}-y^8F_{4½}$
5168.17	−2	1?	IV A	19343.84	$b^8D^\circ_{4½}-157_{4½}$
5167.20	−1	40	III	19347.47	$b^8D^\circ_{3½}-y^8F_{3½}$
5166.70	−2	1200	II	19349.34	$a^{10}D^\circ_{3½}-y^{10}P_{3½}$
5164.53	0	1	V ?	19357.47	$b^8D^\circ_{1½}-y^6P_{1½}$
5163.03	+1	1	V	19363.09	$b^8D^\circ_{2½}-x^6P_{3½}$
5162.39	+1	3	IV ?	19365.49	$b^8D^\circ_{2½}-y^8F_{1½}$
5160.07	{0 −19}	2000	II	19374.20	{$a^{10}D^\circ_{4½}-y^{10}P_{4½}$ $(a^8D^\circ_{5½}-z^6F_{5½})$}
5157.22	+3	1	IV	19384.91	$b^8D^\circ_{1½}-x^6P_{2½}$
5156.41	{+2 −1}	30	III	19387.95	{$y^8P_{3½}-10^\circ_{3½,\,4½}$ $(b^8D^\circ_{2½}-y^6P_{1½})$}
5155.42	+1	125	III	19391.67	$z^8P_{2½}-f^8D^\circ_{1½}$
5152.26	−2	1	V	19403.57	$a^6D^\circ_{4½}-154_{3½}$
5150.80	−1	150	III	19409.07	$b^8D^\circ_{4½}-y^8F_{5½}$
5149.30	+1	10	III	19414.72	$b^8D^\circ_{3½}-x^6P_{3½}$
5148.41	+1	8	III A	19418.08	$a^8D^\circ_{1½}-w^8P_{2½}$
5147.80	0	30	III	19420.38	$b^8D^\circ_{4½}-y^8F_{4½}$
5146.40	−1	8	{III V E}	19425.66	$z^8P_{3½}-f^8D^\circ_{3½}$
5145.70	0	2	IV	19428.30	$b^8D^\circ_{4½}-y^8F_{3½}$
5141.06	0	40	III	19445.84	$z^8P_{3½}-f^8D^\circ_{2½}$
5138.51	+2	3	IV	19455.49	$a^6D^\circ_{2½}-y^6P_{3½}$
5137.52	−4	4	IV A?	19459.24	$a^{10}D^\circ_{4½}-118_{3½}$
5135.91	+3	1	V	19465.34	$a^6D^\circ_{4½}-155_{3½}$
5135.44	−1	8	IV	19467.12	$b^8D^\circ_{3½}-x^6P_{2½}$
5133.52	+2	1500	II	19474.40	$a^{10}D^\circ_{2½}-y^{10}P_{3½}$
5132.42	+.1	10	IV A?	19478.57	$a^6D^\circ_{1½}-y^6P_{2½}$
5130.83	0	3	IV	19484.61	$b^8D^\circ_{1½}-x^6P_{1½}$
5130.47	−3	3	V ?	19485.98	$z^8P_{4½}-g^8D^\circ_{3½}$
5130.08	−1	200	III	19487.46	$z^8P_{2½}-f^8D^\circ_{3½}$
5129.10	+1	1200	II	19491.18	$a^{10}D^\circ_{5½}-y^{10}P_{5½}$
5127.93	0	25	III	19495.63	$b^8D^\circ_{4½}-x^6P_{3½}$
5124.77	−1	300	III	19507.65	$z^8P_{2½}-f^8D^\circ_{2½}$
5122.85	+2	1	V	19514.96	$b^8D^\circ_{2½}-x^6P_{1½}$
5119.47	0	20	III	19527.85	$b^8D^\circ_{5½}-157_{4½}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
5118.88	+ 2	2	IV A	19530.10	$a^{6}D^{\circ}_{\frac{1}{2}}-y^{8}F_{1\frac{1}{2}}$
5118.15	+ 2	1	V	19532.88	$a^{6}D^{\circ}_{\frac{1}{2}}-y^{8}F_{\frac{1}{2}}$
5115.74	0	4	IV ?	19542.08	$a^{6}D^{\circ}_{1\frac{1}{2}}-y^{8}F_{2\frac{1}{2}}$
5114.37	0	800	II	19547.32	$a^{10}D^{\circ}_{3\frac{1}{2}}-y^{10}P_{4\frac{1}{2}}$
5113.0	− 2	5	IV	19552.6	$a^{6}D^{\circ}_{\frac{1}{2}}-y^{6}P_{1\frac{1}{2}}$
5112.85	0	6	IV ?	19553.13	$a^{6}D^{\circ}_{3\frac{1}{2}}-y^{6}P_{3\frac{1}{2}}$
5110.99	0	30	IV	19560.25	$a^{6}D^{\circ}_{2\frac{1}{2}}-y^{6}P_{2\frac{1}{2}}$
5103.65	+ 1	1	V ?	19588.38	$a^{6}D^{\circ}_{1\frac{1}{2}}-y^{8}F_{\frac{1}{2}}$
5102.43	+ 2	12	IV	19593.06	$b^{8}D^{\circ}_{5\frac{1}{2}}-y^{8}F_{5\frac{1}{2}}$
5101.27	0	6	IV	19597.51	$a^{6}D^{\circ}_{2\frac{1}{2}}-y^{8}F_{3\frac{1}{2}}$
5098.73	− 2	300	III	19607.28	$z^{8}P_{4\frac{1}{2}}-g^{8}D^{\circ}_{4\frac{1}{2}}$
5098.55	+ 1	40	III	19607.97	$a^{6}D^{\circ}_{1\frac{1}{2}}-y^{6}P_{1\frac{1}{2}}$
5096.44	0	300	III	19616.09	$b^{8}D^{\circ}_{5\frac{1}{2}}-y^{8}F_{6\frac{1}{2}}$
5094.44	− 2	3	V E	19623.79	$(a^{6}D^{\circ}_{2\frac{1}{2}}-y^{8}F_{2\frac{1}{2}})$
5092.69	+ 1	600	III	19630.53	$z^{8}P_{4\frac{1}{2}}-g^{8}D^{\circ}_{5\frac{1}{2}}$
5092.30	+ 5	3	III A	19632.04	$a^{10}D^{\circ}_{3\frac{1}{2}}-118_{3\frac{1}{2}}$
5091.39	0	3	IV A	19635.54	$a^{6}D^{\circ}_{1\frac{1}{2}}-x^{6}P_{2\frac{1}{2}}$
5089.10	0	250	III	19644.38	$a^{6}D^{\circ}_{4\frac{1}{2}}-y^{6}P_{3\frac{1}{2}}$
5087.18	0	4	IV	19651.79	$a^{8}D^{\circ}_{5\frac{1}{2}}-z^{6}F_{4\frac{1}{2}}$
5085.62	0	30	III	19657.82	$a^{6}D^{\circ}_{3\frac{1}{2}}-y^{6}P_{2\frac{1}{2}}$
5083.81	+ 1	1	V	19664.82	$a^{6}D^{\circ}_{2\frac{1}{2}}-x^{6}P_{3\frac{1}{2}}$
5079.97	− 1	5	IV A	19679.69	$a^{6}D^{\circ}_{\frac{1}{2}}-x^{6}P_{1\frac{1}{2}}$
5079.25	− 1	1	IV	19682.47	$y^{8}P_{4\frac{1}{2}}-f^{10}P^{\circ}_{5\frac{1}{2}}$
5078.05	+ 1	15	III	19687.13	$a^{6}D^{\circ}_{3\frac{1}{2}}-y^{8}F_{4\frac{1}{2}}$
5077.41	+ 1	30	III	19689.61	$a^{6}D^{\circ}_{2\frac{1}{2}}-y^{6}P_{1\frac{1}{2}}$
5076.01	+ 1	1	IV	19695.04	$a^{6}D^{\circ}_{3\frac{1}{2}}-y^{8}F_{3\frac{1}{2}}$
5070.30	− 1	8	III	19717.22	$a^{6}D^{\circ}_{2\frac{1}{2}}-x^{6}P_{2\frac{1}{2}}$
5069.26	+ 1	2	IV	19721.26	$a^{6}D^{\circ}_{3\frac{1}{2}}-y^{8}F_{2\frac{1}{2}}$
5067.95	0	400	III	19726.36	$a^{10}D^{\circ}_{4\frac{1}{2}}-y^{10}P_{5\frac{1}{2}}$
5065.72	+ 3	3	IV	19735.04	$a^{6}D^{\circ}_{1\frac{1}{2}}-x^{6}P_{1\frac{1}{2}}$
5063.74	+ 1	200	III	19742.76	$z^{10}P_{4\frac{1}{2}}-4^{\circ}$
5060.00	+ 2	5	III A	19757.35	$a^{10}D^{\circ}_{2\frac{1}{2}}-118_{3\frac{1}{2}}$
5058.71	0	2	IV A	19762.39	$a^{6}D^{\circ}_{3\frac{1}{2}}-x^{6}P_{3\frac{1}{2}}$
5057.50	− 1	15	III	19767.12	$a^{6}D^{\circ}_{4\frac{1}{2}}-y^{8}F_{5\frac{1}{2}}$
5056.02	+ 1	10	III A	19772.91	$a^{8}D^{\circ}_{4\frac{1}{2}}-z^{6}F_{5\frac{1}{2}}$
5052.61	+ 2	4	IV A	19786.25	$a^{6}D^{\circ}_{4\frac{1}{2}}-y^{8}F_{3\frac{1}{2}}$
5045.34	− 1	10	IV A	19814.76	$a^{6}D^{\circ}_{3\frac{1}{2}}-x^{6}P_{2\frac{1}{2}}$
5044.82	− 1	1 −	V	19816.80	$a^{6}D^{\circ}_{2\frac{1}{2}}-x^{6}P_{1\frac{1}{2}}$
5035.44	− 2	20	IV	19853.72	$a^{6}D^{\circ}_{4\frac{1}{2}}-x^{6}P_{3\frac{1}{2}}$
5033.55	0	500	III	19861.17	$z^{8}P_{3\frac{1}{2}}-f^{8}D^{\circ}_{4\frac{1}{2}}$
5029.54	− 2	600?	III	19877.01	$z^{10}P_{4\frac{1}{2}}-e^{10}D^{\circ}_{3\frac{1}{2}}$
5029.4	− 1	100?	III	19877.6	$z^{6}P_{2\frac{1}{2}}-e^{6}P^{\circ}_{3\frac{1}{2}}$
5022.91	0	2000	III	19903.24	$z^{10}P_{4\frac{1}{2}}-e^{10}D^{\circ}_{4\frac{1}{2}}$
5015.64	0	5	IV A	19932.09	$a^{8}D^{\circ}_{4\frac{1}{2}}-123_{3\frac{1}{2}}$
5013.17	+ 2	1500	III	19941.91	$z^{10}P_{4\frac{1}{2}}-e^{10}D^{\circ}_{5\frac{1}{2}}$
5009.93	+ 1	2	IV	19954.81	$z^{8}P_{4\frac{1}{2}}-e^{6}D^{\circ}_{4\frac{1}{2}}$
4986.79	+ 3	60	III	20047.40	$z^{8}P_{4\frac{1}{2}}-g^{8}S^{\circ}_{3\frac{1}{2}}$
4984.02	+ 1	12	III A	20058.54	$z^{10}P_{3\frac{1}{2}}-3^{\circ}_{3\frac{1}{2},\,4\frac{1}{2}}$
4975.76	0	300	III	20091.84	$z^{6}P_{1\frac{1}{2}}-e^{6}P^{\circ}_{2\frac{1}{2}}$
4968.73	+ 1	150	III	20120.27	$z^{8}P_{3\frac{1}{2}}-g^{8}D^{\circ}_{2\frac{1}{2}}$
4962.55	0	500	III	20145.32	$z^{8}P_{3\frac{1}{2}}-g^{8}D^{\circ}_{3\frac{1}{2}}$
4960.21	+ 1	400	III	20154.83	$z^{8}P_{2\frac{1}{2}}-g^{8}D^{\circ}_{1\frac{1}{2}}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
4953.52	+1	300	III	20182.05	$z^{8}P_{2\frac{1}{2}}-g^{8}D^{\circ}_{2\frac{1}{2}}$
4951.32	+2	1	IV A	20191.02	$a^{8}D^{\circ}_{3\frac{1}{2}}-123_{3\frac{1}{2}}$
4947.39	+1	200	III	20207.05	$z^{8}P_{2\frac{1}{2}}-g^{8}D^{\circ}_{3\frac{1}{2}}$
4938.31	+3	300	III	20244.21	$z^{6}P_{3\frac{1}{2}}-e^{6}P^{\circ}_{3\frac{1}{2}}$
4932.83	0	125	III	20266.70	$z^{8}P_{3\frac{1}{2}}-g^{8}D^{\circ}_{4\frac{1}{2}}$
4929.06	+2	1	IV A	20282.20	$a^{8}D^{\circ}_{3\frac{1}{2}}-124_{2\frac{1}{2}}$
4928.02	+2	80	III	20286.48	$z^{6}P_{2\frac{1}{2}}-h^{8}S^{\circ}_{3\frac{1}{2}}$
4924.73	+3	80‖	III	20300.03	$z^{6}P_{1\frac{1}{2}}-e^{6}P^{\circ}_{1\frac{1}{2}}$
4922.25	+1	8	III A	20310.26	$a^{8}D^{\circ}_{3\frac{1}{2}}-z^{6}F_{4\frac{1}{2}}$
4917.50	0	5	IV	20329.9	$z^{6}P_{2\frac{1}{2}}-e^{6}P^{\circ}_{2\frac{1}{2}}$
4911.40	−1	1800	III	20355.13	$z^{10}P_{3\frac{1}{2}}-e^{10}D^{\circ}_{2\frac{1}{2}}$
4907.18	0	1500	III	20372.63	$z^{10}P_{3\frac{1}{2}}-e^{10}D^{\circ}_{3\frac{1}{2}}$
4902.66	0	1	IV A	20391.41	$a^{8}D^{\circ}_{4\frac{1}{2}}-129_{3\frac{1}{2},\,4\frac{1}{2}}$
4900.86	+1	600	III	20398.90	$z^{10}P_{3\frac{1}{2}}-e^{10}D^{\circ}_{4\frac{1}{2}}$
4900.49	+4	3	V	20400.44	$a^{8}D^{\circ}_{4\frac{1}{2}}-z^{6}F_{3\frac{1}{2}}$
4898.39	−1	1	V	20409.19	$b^{8}D^{\circ}_{1\frac{1}{2}}-y^{6}F_{1\frac{1}{2}}$
4894.68	+2	150	III	20424.66	$a^{8}D^{\circ}_{5\frac{1}{2}}-u^{8}P_{4\frac{1}{2}}$
4893.5	−4	1−	V	20429.6	$b^{8}D^{\circ}_{1\frac{1}{2}}-x^{8}F_{1\frac{1}{2}}$
4891.02	−1	1	IV	20439.94	$b^{8}D^{\circ}_{2\frac{1}{2}}-y^{6}F_{2\frac{1}{2}}$
4887.45	−2	1	IV A	20454.87	$a^{8}D^{\circ}_{2\frac{1}{2}}-124_{2\frac{1}{2}}$
4884.05	−1	150	III	20469.11	$a^{8}D^{\circ}_{5\frac{1}{2}}-v^{8}P_{4\frac{1}{2}}$
4883.05	+6 −10	1	IV	20473.30	$b^{8}D^{\circ}_{2\frac{1}{2}}-x^{8}F_{2\frac{1}{2}}$ $b^{8}D^{\circ}_{2\frac{1}{2}}-x^{8}F_{3\frac{1}{2}}$
4879.47	−1	4	IV	20488.32	$b^{8}D^{\circ}_{3\frac{1}{2}}-y^{6}F_{4\frac{1}{2}}$
4879.17	+2	6	III A	20489.58	$z^{10}P_{4\frac{1}{2}}-e^{8}P^{\circ}_{3\frac{1}{2}}$
4873.54	+3	1	V	20513.25	$a^{8}D^{\circ}_{2\frac{1}{2}}-125_{1\frac{1}{2}}$
4870.89	+2	1	IV A	20524.41	$b^{8}D^{\circ}_{3\frac{1}{2}}-x^{8}F_{3\frac{1}{2}}$
4867.62	+1 +2 −16	600	III	20538.20	$z^{6}P_{2\frac{1}{2}}-e^{6}P^{\circ}_{1\frac{1}{2}}$ $a^{10}D^{\circ}_{6\frac{1}{2}}-x^{10}P_{5\frac{1}{2}}$ $(b^{8}D^{\circ}_{3\frac{1}{2}}-x^{8}F_{4\frac{1}{2}})$
4866.40	0	4	III A	20543.35	$z^{10}P_{4\frac{1}{2}}-e^{8}D^{\circ}_{4\frac{1}{2}}$
4864.7	−4	1−	V	20550.5	$a^{8}D^{\circ}_{2\frac{1}{2}}-126_{1\frac{1}{2}}$
4860.86	0	12	IV	20566.76	$b^{8}D^{\circ}_{4\frac{1}{2}}-y^{6}F_{5\frac{1}{2}}$
4860.29	0	1	IV	20569.18	$b^{8}D^{\circ}_{4\frac{1}{2}}-y^{6}F_{4\frac{1}{2}}$
4856.78	−2	3	IV A	20584.04	$z^{8}P_{4\frac{1}{2}}-f^{10}S^{\circ}_{4\frac{1}{2}}$
4852.03	0	5	IV	20604.19	$a^{6}D^{\circ}_{\frac{1}{2}}-y^{6}F_{1\frac{1}{2}}$
4851.87	0	3	IV	20604.87	$b^{8}D^{\circ}_{4\frac{1}{2}}-x^{8}F_{5\frac{1}{2}}$
4851.24	0	6	IV	20607.55	$a^{6}D^{\circ}_{\frac{1}{2}}-y^{6}F_{\frac{1}{2}}$
4849.64	0	300	III	20614.35	$z^{8}P_{3\frac{1}{2}}-e^{6}D^{\circ}_{4\frac{1}{2}}$
4848.70	+2	2	IV	20618.34	$b^{8}D^{\circ}_{4\frac{1}{2}}-x^{8}F_{4\frac{1}{2}}$
4844.31	−1 +9	25	III	20637.03	$z^{8}P_{3\frac{1}{2}}-e^{6}D^{\circ}_{3\frac{1}{2}}$ $(z^{10}P_{5\frac{1}{2}}-g^{8}D^{\circ}_{4\frac{1}{2}})$
4843.36	−1	30	III	20641.07	$z^{10}P_{4\frac{1}{2}}-e^{8}D^{\circ}_{5\frac{1}{2}}$
4841.14	−2	1	IV A	20650.54	$a^{8}D^{\circ}_{3\frac{1}{2}}-129_{3\frac{1}{2},\,4\frac{1}{2}}$
4840.47	−2	150	III	20653.40	$z^{6}P_{3\frac{1}{2}}-h^{8}S^{\circ}_{3\frac{1}{2}}$
4839.58	−3	2	IV A	20657.20	$a^{10}D^{\circ}_{4\frac{1}{2}}-119_{3\frac{1}{2}}$
4838.92	−1 −8 −9	20	IV	20660.02	$a^{6}D^{\circ}_{1\frac{1}{2}}-y^{6}F_{2\frac{1}{2}}$ $(a^{6}D^{\circ}_{1\frac{1}{2}}-y^{6}F_{1\frac{1}{2}})$ $(a^{8}D^{\circ}_{3\frac{1}{2}}-z^{6}F_{3\frac{1}{2}})$
4838.8	+6	1	V	20660.5	$z^{10}P_{5\frac{1}{2}}-g^{8}D^{\circ}_{5\frac{1}{2}}$
4838.22	0	3	IV V E	20663.00	$a^{6}D^{\circ}_{1\frac{1}{2}}-y^{6}F_{\frac{1}{2}}$

Blend.

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
4835.61	+1	1	V	20674.16	$a^6D^\circ_{1\frac12}-x^8F_{\frac12}$
4832.68	+1	6	IV A	20686.69	$a^{10}D^\circ_{4\frac12}-120_{3\frac12}$
4830.33	−1	250	III	20696.75	$z^6P_{3\frac12}-e^6P^\circ_{2\frac12}$
4829.86	0	20	III	20698.77	$z^8P_{2\frac12}-e^6D^\circ_{3\frac12}$
4829.30	0	200	III	20701.17	$a^8D^\circ_{4\frac12}-v^8P_{3\frac12}$
4827.95	+1	4	IV	20706.96	$z^8P_{3\frac12}-g^8S^\circ_{3\frac12}$
4825.63	−1	40	III	20716.91	$z^{10}P_{4\frac12}-e^8P^\circ_{4\frac12}$
4825.40	0	10	IV	20717.90	$z^8P_{2\frac12}-e^6D^\circ_{2\frac12}$
4822.20	−4	2	IV A	20731.65	$z^8P_{2\frac12}-e^6D^\circ_{1\frac12}$
4820.80	−2	4	III A	20737.67	$a^{10}D^\circ_{3\frac12}-x^8P_{2\frac12}$
4820.47	0	50	IV	20739.09	$a^6D^\circ_{2\frac12}-y^6F_{3\frac12}$
4819.88	0 −7	15	IV	20741.63	$a^6D^\circ_{2\frac12}-y^6F_{2\frac12}$ $(a^6D^\circ_{2\frac12}-y^6F_{1\frac12})$
4815.21	−3	1	V	20761.74	$a^6D^\circ_{2\frac12}-x^8F_{1\frac12}$
4813.55	−3	8	IV A	20768.90	$z^8P_{2\frac12}-g^8S^\circ_{3\frac12}$
4812.07	0	1−	V	20775.29	$a^6D^\circ_{2\frac12}-x^8F_{2\frac12}$
4809.29	0	200	III	20787.30	$a^8D^\circ_{5\frac12}-y^8D_{5\frac12}$
4806.94	0	10	IV	20797.46	$b^8D^\circ_{5\frac12}-x^8F_{6\frac12}$
4805.46	−1 −2	4	IV	20803.86	$a^8D^\circ_{2\frac12}-128_{2\frac12}$ $(a^8D^\circ_{1\frac12}-127_{1\frac12,\,2\frac12})$
4804.08	−1	100	III	20809.84	$a^8D^\circ_{5\frac12}-y^8D_{4\frac12}$
4800.79	+3	15	III	20824.10	$a^8D^\circ_{4\frac12}-u^8P_{4\frac12}$
4799.38	0	12	III	20830.22	$a^{10}D^\circ_{3\frac12}-119_{3\frac12}$
4798.92	−2	15	IV	20832.22	$a^8D^\circ_{2\frac12}-z^6F_{3\frac12}$
4798.06	0	60	III	20835.95	$a^6D^\circ_{3\frac12}-y^6F_{4\frac12}$
4797.9	+1	50?	III	20836.6	$a^6D^\circ_{3\frac12}-y^6F_{3\frac12}$
4797.33	+1	1	V	20839.12	$a^6D^\circ_{3\frac12}-y^6F_{2\frac12}$
4792.58	0 +2	300	III	20859.78	$a^{10}D^\circ_{5\frac12}-x^{10}P_{5\frac12}$ $(a^{10}D^\circ_{3\frac12}-120_{3\frac12})$
4791.84†	−6 +12	10	III A	20862.99	$a^{10}D^\circ_{2\frac12}-x^8P_{2\frac12}$ $a^8D^\circ_{3\frac12}-z^6F_{2\frac12}$
4790.6	+3	1−	V	20868.4	$a^8D^\circ_{4\frac12}-v^8P_{4\frac12}$
4789.62	+4	2	V	20872.67	$a^6D^\circ_{3\frac12}-x^8F_{2\frac12}$?
4787.98	−2	4	IV A	20879.82	$a^{10}D^\circ_{4\frac12}-x^8P_{3\frac12}$
4784.01	+1	40	III	20897.14	$z^{10}P_{4\frac12}-f^8D^\circ_{5\frac12}$
4781.32	0	50	III	20908.90	$a^{10}D^\circ_{5\frac12}-w^8P_{4\frac12}$
4779.70	0	4	IV A	20915.99	$a^{10}D^\circ_{3\frac12}-121_{2\frac12}$
4778.64	−2	100	III	20920.62	$a^8D^\circ_{4\frac12}-u^8P_{3\frac12}$
4777.70	0	200	III	20924.74	$a^6D^\circ_{4\frac12}-y^6F_{5\frac12}$
4777.16	+2	6	IV	20927.11	$a^6D^\circ_{4\frac12}-y^6F_{4\frac12}$
4770.78	0	150	III	20955.09	$a^8D^\circ_{5\frac12}-132_{5\frac12}$
4769.61	−1	10	III	20960.23	$a^8D^\circ_{3\frac12}-v^8P_{3\frac12}$
4768.28	+1	3	IV	20966.08	$z^{10}P_{3\frac12}-e^8D^\circ_{3\frac12}$
4766.65	−1	4	IV	20973.25	$z^8P_{4\frac12}-e^6P^\circ_{3\frac12}$
4763.97	0 +7	125	III	20985.05	$a^{10}D^\circ_{2\frac12}-120_{3\frac12}$ $(z^{10}P_{3\frac12}-e^8P^\circ_{3\frac12})$
4763.24	0	10	IV	20988.26	$z^6P_{1\frac12}-f^6S^\circ_{2\frac12}$
4762.92	0	40	III	20989.67	$a^8D^\circ_{3\frac12}-v^8P_{2\frac12}$
4755.93	0 −11	40	III	21020.52	$a^8D^\circ_{3\frac12}-u^8P_{2\frac12}$ $(a^8D^\circ_{4\frac12}-y^8D_{3\frac12})$
4754.05	0	2	V	21028.83	$a^6D^\circ_{4\frac12}-160_{4\frac12,\,5\frac12}$
4752.42	−1	2	IV	21036.05	$a^8D^\circ_{2\frac12}-z^6F_{2\frac12}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
4750.08	−2	2	IV	21046.41	$a^{8}D^{\circ}_{5½}-134_{4½}$
4748.5	−12	1	V	21053.4	$a^{10}D^{\circ}_{3½}-x^{8}P_{3½}$?
4741.76	−2	8	IV	21083.34	$a^{8}D^{\circ}_{3½}-u^{8}P_{4½}$
4740.50	−2	200	III	21088.94	$a^{10}D^{\circ}_{5½}-x^{10}P_{4½}$
4739.16	+1	50	III	21094.90	$a^{10}D^{\circ}_{4½}-x^{10}P_{5½}$
4736.58	−1	40	III	21106.39	$z^{10}P_{3½}-e^{8}P^{\circ}_{2½}$
4731.82	−1	3	IV A	21127.63	$a^{8}D^{\circ}_{3½}-v^{8}P_{4½}$
4730.70	+1	4	V ?	21132.63	$a^{8}D^{\circ}_{2½}-v^{8}P_{3½}$
4728.13	0	30	III	21144.11	$a^{10}D^{\circ}_{4½}-w^{8}P_{4½}$
4727.47	−1	6	IV	21147.07	$a^{8}D^{\circ}_{1½}-z^{6}F_{2½}$
4724.08	−2	20	III	21162.24	$a^{8}D^{\circ}_{2½}-v^{8}P_{2½}$
4723.90	−1	15	III	21163.05	$a^{8}D^{\circ}_{3½}-y^{8}D_{2½}$
4720.54	−1	8	III A	21178.11	$a^{10}D^{\circ}_{2½}-x^{8}P_{3½}$
4720.21	0	6	IV	21179.59	$a^{8}D^{\circ}_{3½}-u^{8}P_{3½}$
4718.61	0	40	III	21186.77	$a^{8}D^{\circ}_{4½}-y^{8}D_{5½}$
4717.22	0	40	III	21193.02	$a^{8}D^{\circ}_{2½}-u^{8}P_{2½}$
4713.59	−1	300	III	21209.34	$a^{8}D^{\circ}_{4½}-y^{8}D_{4½}$
4709.81	−1	20	III	21226.36	$z^{6}P_{2½}-f^{6}S^{\circ}_{2½}$
4705.99†	−3	3	IV ?	21243.59	$z^{8}P_{3½}-f^{10}S^{\circ}_{4½}$?
4704.59	0	20	III	21249.91	$z^{10}P_{4½}-f^{8}D^{\circ}_{4½}$
4703.92	−1	6	III	21252.93	$a^{8}D^{\circ}_{2½}-z^{6}F_{1½}$
4700.45†	−3	3	IV	21268.63	$a^{10}D^{\circ}_{5½}-x^{8}P_{4½}$?
4699.42	−2	5	IV	21273.29	$a^{8}D^{\circ}_{1½}-v^{8}P_{2½}$
4698.12	−2	150	III	21279.17	$a^{8}D^{\circ}_{3½}-y^{8}D_{3½}$
4697.59	0	8	III	21281.57	$a^{10}D^{\circ}_{5½}-122_{4½}$
4692.64	+1	40	III	21304.02	$a^{8}D^{\circ}_{1½}-u^{8}P_{2½}$
4691.30	−1	3	V	21310.11	$z^{10}P_{3½}-f^{8}D^{\circ}_{3½}$
4689.74	+1	4	III A	21317.20	$a^{10}D^{\circ}_{3½}-w^{8}P_{4½}$
4688.24	0	200	III	21324.02	$a^{10}D^{\circ}_{4½}-x^{10}P_{4½}$
4686.85	−1	3	IV	21330.34	$z^{10}P_{3½}-f^{8}D^{\circ}_{2½}$
4685.71	−1	3	IV	21335.53	$a^{8}D^{\circ}_{2½}-y^{8}D_{2½}$
4685.25	−1	40	III	21337.62	$a^{8}D^{\circ}_{2½}-y^{8}D_{1½}$
4684.78	−2	4	IV	21339.77	$a^{10}D^{\circ}_{4½}-w^{8}P_{3½}$
4682.1	+2	1−	V	21352.0	$a^{8}D^{\circ}_{2½}-u^{8}P_{3½}$
4681.53	−1	10	III	21354.58	$a^{8}D^{\circ}_{4½}-132_{5½}$
4679.48	0	4	IV	21363.94	$a^{8}D^{\circ}_{1½}-z^{6}F_{1½}$
4675.49	0	20	III	21382.17	$z^{8}P_{4½}-h^{8}S^{\circ}_{3½}$
4661.88	0	7000R	I	21444.59	$a^{8}S^{\circ}_{3½}-y^{8}P_{2½}$
4661.46	0	10	IV	21446.52	$a^{8}D^{\circ}_{1½}-y^{8}D_{2½}$
4661.01	+1 +1	15	IV	21448.59	$a^{8}D^{\circ}_{1½}-y^{8}D_{1½}$ ($a^{8}D^{\circ}_{1½}-z^{6}F_{½}$)
4660.36	0	100	III	21451.58	$a^{8}D^{\circ}_{2½}-y^{8}D_{3½}$
4656.73	0	60	III	21468.30	$a^{8}D^{\circ}_{3½}-y^{8}D_{4½}$
4650.48	0	50	III	21497.16	$a^{10}D^{\circ}_{3½}-x^{10}P_{4½}$
4649.06	−1	8	III	21503.72	$a^{10}D^{\circ}_{4½}-x^{8}P_{4½}$
4648.4	+3	1−	V	21506.8	$a^{10}D^{\circ}_{3½}-w^{8}P_{2½}$
4647.41	0	15	IV	21511.36	$a^{8}D^{\circ}_{5½}-141_{4½}$
4647.1	+1	1−	V	21512.8	$a^{10}D^{\circ}_{3½}-w^{8}P_{3½}$
4646.25	+1	3	V	21516.73	$a^{10}D^{\circ}_{4½}-122_{4½}$
4642.50	−1	15	IV	21534.11	$z^{10}P_{4½}-g^{8}D^{\circ}_{3½}$
4642.26	−2	4	IV	21535.22	$z^{6}P_{2½}-i^{8}S^{\circ}_{3½}$
4629.82	0	15	IV	21593.08	$z^{6}P_{3½}-f^{6}S^{\circ}_{2½}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
4627.22	−1	8000R	I	21605.22	$a^8S^\circ_{3\frac{1}{2}} - y^8P_{3\frac{1}{2}}$
4625.30	0	50	III	21614.18	$z^{10}P_{5\frac{1}{2}} - f^{10}S^\circ_{4\frac{1}{2}}$
4621.34	−1	15	IV	21632.71	$z^8P_{3\frac{1}{2}} - e^6P^\circ_{3\frac{1}{2}}$
4620.2	0	1	V	21638.0	$a^{10}D^\circ_{2\frac{1}{2}} - w^8P_{3\frac{1}{2}}$
4616.49	0	30	III	21655.43	$z^{10}P_{4\frac{1}{2}} - g^8D^\circ_{4\frac{1}{2}}$
4611.9	−4	1−	IV ?	21677.0	$a^{10}D^\circ_{3\frac{1}{2}} - x^8P_{4\frac{1}{2}}$
4611.51	−1	50	III	21678.82	$z^{10}P_{4\frac{1}{2}} - g^8D^\circ_{5\frac{1}{2}}$
4609.14	−1	1	V	21689.96	$a^{10}D^\circ_{3\frac{1}{2}} - 122_{4\frac{1}{2}}$
4608.18	−1	2	IV	21694.48	$z^8P_{2\frac{1}{2}} - e^6P^\circ_{3\frac{1}{2}}$
4605.85	+2	8	IV	21705.46	$y^8P_{4\frac{1}{2}} - 18^\circ$
4602.63	0	15	IV	21720.64	$a^8D^\circ_{5\frac{1}{2}} - 142_{5\frac{1}{2}}$
4599.19	+1	6	V	21736.89	$b^8D^\circ_{5\frac{1}{2}} - x^{10}F_{6\frac{1}{2}}$
4597.34	−1	40	III	21745.64	$z^{10}P_{3\frac{1}{2}} - f^8D^\circ_{4\frac{1}{2}}$
4594.03	−1	10,000R	I	21761.30	$a^8S^\circ_{3\frac{1}{2}} - y^8P_{4\frac{1}{2}}$
4580.75	0	6	IV	21824.39	$a^8D^\circ_{4\frac{1}{2}} - 138_{3\frac{1}{2}}$
4579.77	−2	3	V	21829.06	$a^8D^\circ_{5\frac{1}{2}} - 145_{4\frac{1}{2}}$
4565.42	+2	4	V	21897.67	$a^6D^\circ_{4\frac{1}{2}} - x^{10}F_{4\frac{1}{2}}$?
4564.53	−1	15	IV	21901.94	$z^6P_{3\frac{1}{2}} - i^8S^\circ_{3\frac{1}{2}}$
4562.68	0	10	IV	21910.82	$a^8D^\circ_{4\frac{1}{2}} - 141_{4\frac{1}{2}}$
4543.18	−2	10	IV	22004.86	$z^{10}P_{3\frac{1}{2}} - g^8D^\circ_{2\frac{1}{2}}$
4540.59	+2	4	IV	22017.42	$a^8D^\circ_{2\frac{1}{2}} - 135_{2\frac{1}{2}}$
4538.05	+1	20	III	22029.74	$z^{10}P_{3\frac{1}{2}} - g^8D^\circ_{3\frac{1}{2}}$
4535.59	0	200	III	22041.69	$z^8P_{3\frac{1}{2}} - h^8S^\circ_{3\frac{1}{2}}$
4527.01	−1	3	IV	22083.46	$a^8D^\circ_{3\frac{1}{2}} - 138_{3\frac{1}{2}}$
4526.69	0	60	III	22085.02	$z^8P_{3\frac{1}{2}} - e^6P^\circ_{2\frac{1}{2}}$
4524.49	0	12	IV	22095.76	$z^{10}P_{4\frac{1}{2}} - g^8S^\circ_{3\frac{1}{2}}$
4522.9	−1	[25]	III	22103.5	$z^8P_{2\frac{1}{2}} - h^8S^\circ_{3\frac{1}{2}}$
4517.76	−2	20	IV	22128.68	$a^8D^\circ_{1\frac{1}{2}} - 135_{2\frac{1}{2}}$
4514.06	0	15	IV	22146.82	$z^8P_{2\frac{1}{2}} - e^6P^\circ_{2\frac{1}{2}}$
4513.20	+2	20	III	22151.04	$z^{10}P_{3\frac{1}{2}} - g^8D^\circ_{4\frac{1}{2}}$
4509.04	−2	4	IV	22171.47	$a^8D^\circ_{4\frac{1}{2}} - 144_{3\frac{1}{2}}$
4497.45	−3	4	{V E IV}	22228.61	$a^8D^\circ_{4\frac{1}{2}} - 145_{4\frac{1}{2}}$
4492.39	0	4	V	22253.64	$a^8D^\circ_{2\frac{1}{2}} - 137_{2\frac{1}{2},\,3\frac{1}{2}}$
4471.99	−1	50	III	22355.16	$z^8P_{2\frac{1}{2}} - e^6P^\circ_{1\frac{1}{2}}$
4464.563	0	40	III	22392.34	$a^8D^\circ_{4\frac{1}{2}} - 146_{3\frac{1}{2}}$
4456.98	−1	5	IV	22430.44	$a^8D^\circ_{3\frac{1}{2}} - 144_{3\frac{1}{2}}$
4427.0	+5	5	IV	22582.3	$a^8D^\circ_{4\frac{1}{2}} - 147_{4\frac{1}{2}}$
4422.96	−1	3	IV	22602.97	$a^8D^\circ_{2\frac{1}{2}} - 144_{3\frac{1}{2}}$
4417.55	+2	5	IV	22630.65	$z^8P_{4\frac{1}{2}} - i^8S^\circ_{3\frac{1}{2}}$
4417.25	0	80	III	22632.19	$z^{10}P_{4\frac{1}{2}} - f^{10}S^\circ_{4\frac{1}{2}}$
4413.51	0	12	IV	22651.36	$a^8D^\circ_{3\frac{1}{2}} - 146_{3\frac{1}{2}}$
4387.88	−1	250	III	22783.67	$a^{10}D^\circ_{4\frac{1}{2}} - x^{10}P_{3\frac{1}{2}}$
4380.16	+1	4	IV	22823.82	$a^8D^\circ_{2\frac{1}{2}} - 146_{3\frac{1}{2}}$
4377.33	−3	2	V	22838.58	$a^{10}D^\circ_{2\frac{1}{2}} - 125_{1\frac{1}{2}}$
4370.47	0	80	III	22874.43	$z^8P_{4\frac{1}{2}} - h^8D^\circ_{5\frac{1}{2}}$
4369.47	+1	40	{III V E}	22879.66	$z^8P_{4\frac{1}{2}} - h^8D^\circ_{4\frac{1}{2}}$
4368.52	+2	8§	IV	22884.64	$z^8P_{4\frac{1}{2}} - h^8D^\circ_{3\frac{1}{2}}$
4366.51	+2	6	IV	22895.17	$a^8D^\circ_{5\frac{1}{2}} - 157_{4\frac{1}{2}}$
4354.80	+1	150	III	22956.74	$a^{10}D^\circ_{3\frac{1}{2}} - x^{10}P_{3\frac{1}{2}}$
4350.13	+1	12	IV	22981.38	$z^8P_{3\frac{1}{2}} - f^6S^\circ_{2\frac{1}{2}}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
4349.72	0	25	IV	22983.55	$a^8D^\circ_{5\frac{1}{2}}-y^8F_{6\frac{1}{2}}$
4347.26	−2	5	IV	22996.55	$a^8D^\circ_{4\frac{1}{2}}-154_{3\frac{1}{2}}$
4345.91	0	100	III	23003.70	$a^{10}D^\circ_{3\frac{1}{2}}-128_{2\frac{1}{2}}$
4343.25	0	30	III	23017.78	$a^{10}D^\circ_{2\frac{1}{2}}-127_{1\frac{1}{2},\,2\frac{1}{2}}$
4342.55	−2	4	IV	23021.49	$z^{10}P_{4\frac{1}{2}}-e^6P^\circ_{3\frac{1}{2}}$
4338.46	0	5	IV	23043.20	$z^8P_{2\frac{1}{2}}-f^6S^\circ_{2\frac{1}{2}}$
4337.68	−1	200	III	23047.34	$a^{10}D^\circ_{5\frac{1}{2}}-u^8P_{4\frac{1}{2}}$
4331.175	0	100	III	23081.95	$a^{10}D^\circ_{2\frac{1}{2}}-x^{10}P_{3\frac{1}{2}}$
4329.97	0	200	III	23088.38	$a^{10}D^\circ_{6\frac{1}{2}}-y^8D_{5\frac{1}{2}}$
4329.36	0	200	III	23091.63	$a^{10}D^\circ_{5\frac{1}{2}}-v^8P_{4\frac{1}{2}}$
4322.57	0	100	III	23127.90	$z^{10}P_{3\frac{1}{2}}-f^{10}S^\circ_{4\frac{1}{2}}$
4316.7	0	1	IV	23159.4	$a^{10}D^\circ_{4\frac{1}{2}}-v^8P_{3\frac{1}{2}}$
4314.3	−1	3	IV	23172.2	$a^8D^\circ_{4\frac{1}{2}}-156_{5\frac{1}{2}}$
4298.73	0 −8	300	III	23256.16	$a^{10}D^\circ_{6\frac{1}{2}}-132_{5\frac{1}{2}}$ $(z^6P_{1\frac{1}{2}}-f^6D^\circ_{2\frac{1}{2}})$
4297.43	−2	25	III	23263.20	$z^6P_{1\frac{1}{2}}-f^6D^\circ_{1\frac{1}{2}}$
4293.87	−1	25	III	23282.49	$a^{10}D^\circ_{4\frac{1}{2}}-u^8P_{4\frac{1}{2}}$
4292.42	−2	[4]	IV	23290.35	$z^8P_{3\frac{1}{2}}-i^8S^\circ_{3\frac{1}{2}}$
4287.81	−1	3	IV	23315.39	$z^8P_{3\frac{1}{2}}-f^{10}D^\circ_{2\frac{1}{2}}$
4287.44	+1	20	III	23317.40	$a^8D^\circ_{3\frac{1}{2}}-155_{3\frac{1}{2}}$
4285.72	+1	10	III	23326.76	$a^{10}D^\circ_{4\frac{1}{2}}-v^8P_{4\frac{1}{2}}$
4284.66	0	20	III	23332.53	$a^{10}D^\circ_{3\frac{1}{2}}-v^8P_{3\frac{1}{2}}$
4281.08	0	5	IV	23352.04	$z^8P_{2\frac{1}{2}}-i^8S^\circ_{3\frac{1}{2}}$
4279.62	0	10	IV	23360.01	$a^8D^\circ_{4\frac{1}{2}}-y^8F_{5\frac{1}{2}}$
4279.25	0	8	III	23362.03	$a^{10}D^\circ_{3\frac{1}{2}}-v^8P_{2\frac{1}{2}}$
4276.20	+2	30	III V E	23378.69	$a^{10}D^\circ_{4\frac{1}{2}}-u^8P_{3\frac{1}{2}}$
4273.60	−1	3	IV	23392.92	$a^{10}D^\circ_{3\frac{1}{2}}-u^8P_{2\frac{1}{2}}$
4270.50	−1	12	V E IV	23409.90	$a^{10}D^\circ_{5\frac{1}{2}}-y^8D_{5\frac{1}{2}}$
4266.76	0	8	IV	23430.42	$z^{10}P_{4\frac{1}{2}}-h^8S^\circ_{3\frac{1}{2}}$
4266.38	−3	4	V E? IV	23432.50	$a^{10}D^\circ_{5\frac{1}{2}}-y^8D_{4\frac{1}{2}}$
4263.80	−1	5	IV V E	23446.68	$a^8D^\circ_{4\frac{1}{2}}-x^6P_{3\frac{1}{2}}$
4262.17	−1	6	III	23455.65	$a^{10}D^\circ_{3\frac{1}{2}}-u^8P_{4\frac{1}{2}}$
4261.79	0	12	III	23457.74	$a^{10}D^\circ_{2\frac{1}{2}}-v^8P_{3\frac{1}{2}}$
4258.07	+1 −5	30	III	23478.23	$a^{10}D^\circ_{4\frac{1}{2}}-y^8D_{3\frac{1}{2}}$ $(z^6P_{2\frac{1}{2}}-f^6D^\circ_{3\frac{1}{2}})$
4256.4	−4	1	V	23487.4	$a^{10}D^\circ_{2\frac{1}{2}}-v^8P_{2\frac{1}{2}}$
4255.95	0	8	III	23489.93	$a^8D^\circ_{2\frac{1}{2}}-155_{3\frac{1}{2}}$
4255.25	0	12	III	23493.79	$z^6P_{2\frac{1}{2}}-f^6D^\circ_{2\frac{1}{2}}$
4253.93	+2	[4]	IV	23501.08	$z^6P_{2\frac{1}{2}}-f^6D^\circ_{1\frac{1}{2}}$
4247.73	−1	6	IV A	23535.38	$a^{10}D^\circ_{3\frac{1}{2}}-y^8D_{2\frac{1}{2}}$
4247.06	+2	25	III V E	23539.09	$z^8P_{3\frac{1}{2}}-h^8D^\circ_{4\frac{1}{2}}$
4246.13	0	12	III	23544.25	$z^8P_{3\frac{1}{2}}-h^8D^\circ_{3\frac{1}{2}}$
4244.74	−1	50	II	23551.96	$a^{10}D^\circ_{3\frac{1}{2}}-u^8P_{3\frac{1}{2}}$
4240.12	0	6	IV	23577.62	$a^{10}D^\circ_{5\frac{1}{2}}-132_{5\frac{1}{2}}$
4235.90	−1	5	IV	23601.11	$a^8D^\circ_{3\frac{1}{2}}-y^6P_{2\frac{1}{2}}$
4235.02	0	6	IV	23606.02	$z^8P_{2\frac{1}{2}}-h^8D^\circ_{3\frac{1}{2}}$
4234.37	+3	10	III	23609.64	$z^8P_{2\frac{1}{2}}-h^8D^\circ_{2\frac{1}{2}}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
4233.60	+1	8	IV	23613.93	$a^{10}D^{\circ}_{6\frac{1}{2}}-136_{5\frac{1}{2},6\frac{1}{2}}$
4230.63	−2	12	V E III	23630.51	$a^{8}D^{\circ}_{3\frac{1}{2}}-y^{8}F_{4\frac{1}{2}}$
4228.07	+3	3	IV V E	23644.82	$a^{10}D^{\circ}_{4\frac{1}{2}}-y^{8}D_{5\frac{1}{2}}$
4226.87	−2	4	V E IV	23651.53	$a^{10}D^{\circ}_{3\frac{1}{2}}-y^{8}D_{3\frac{1}{2}}$
4224.89	0	4	IV	23662.61	$a^{10}D^{\circ}_{2\frac{1}{2}}-y^{8}D_{1\frac{1}{2}}$
4222.31	+1	25	III	23677.07	$a^{10}D^{\circ}_{2\frac{1}{2}}-u^{8}P_{3\frac{1}{2}}$
4220.670	−1	15	IV	23686.27	$z^{8}P_{4\frac{1}{2}}-f^{10}P^{\circ}_{3\frac{1}{2}}$
4213.86	0	8	IV	23724.55	$z^{10}P_{5\frac{1}{2}}-f^{10}D^{\circ}_{4\frac{1}{2}}$
4209.15	0	15	III A	23751.10	$z^{10}P_{5\frac{1}{2}}-f^{10}D^{\circ}_{5\frac{1}{2}}$
4202.69	0	50	II	23787.61	$z^{10}P_{5\frac{1}{2}}-f^{10}D^{\circ}_{6\frac{1}{2}}$
4198.6	0	1	IV	23810.8	$a^{8}D^{\circ}_{2\frac{1}{2}}-y^{8}F_{3\frac{1}{2}}$
4194.49	0	20	III	23834.11	$z^{6}P_{3\frac{1}{2}}-f^{6}D^{\circ}_{4\frac{1}{2}}$
4192.62	0	12	V E IV	23844.74	$z^{6}P_{3\frac{1}{2}}-f^{6}D^{\circ}_{3\frac{1}{2}}$
4182.22	0	80	II	23904.03	$a^{10}D^{\circ}_{4\frac{1}{2}}-134_{4\frac{1}{2}}$
4178.33	−3	6	IV	23926.29	$z^{10}P_{3\frac{1}{2}}-h^{8}S^{\circ}_{3\frac{1}{2}}$
4177.59	−1	15§	IV	23930.52	$a^{8}D^{\circ}_{2\frac{1}{2}}-x^{6}P_{2\frac{1}{2}}$
4176.73	0	25	III	23935.45	$a^{10}D^{\circ}_{5\frac{1}{2}}-136_{5\frac{1}{2},6\frac{1}{2}}$
4171.70	−2	4	IV	23964.31	$z^{8}P_{4\frac{1}{2}}-5^{\circ}_{3\frac{1}{2}}$
4166.94	−1	5	IV	23991.68	$a^{8}D^{\circ}_{1\frac{1}{2}}-y^{8}F_{1\frac{1}{2}}$
4166.47	0	[1]	IV	23994.39	$a^{8}D^{\circ}_{1\frac{1}{2}}-y^{8}F_{\frac{1}{2}}$
4163.06	−1	4	V	24014.05	$a^{8}D^{\circ}_{1\frac{1}{2}}-y^{6}P_{1\frac{1}{2}}$
4161.72	−1	6	IV	24021.78	$a^{10}D^{\circ}_{6\frac{1}{2}}-142_{5\frac{1}{2}}$
4160.27	−1	3	IV	24030.15	$a^{8}D^{\circ}_{2\frac{1}{2}}-x^{6}P_{1\frac{1}{2}}$
4158.78	−1	8	IV	24038.76	$z^{8}P_{4\frac{1}{2}}-7^{\circ}_{3\frac{1}{2}}$
4157.724	0	40	III	24044.86	$a^{10}D^{\circ}_{3\frac{1}{2}}-133_{2\frac{1}{2}}$
4152.14	0	10	IV	24077.20	$a^{10}D^{\circ}_{3\frac{1}{2}}-134_{4\frac{1}{2}}$
4141.14	−1	2	IV	24141.16	$a^{8}D^{\circ}_{1\frac{1}{2}}-x^{6}P_{1\frac{1}{2}}$
4139.22	−1	10	IV	24152.35	$z^{8}P_{4\frac{1}{2}}-8^{\circ}_{4\frac{1}{2},5\frac{1}{2}}$
4138.51	0 −2	8	IV	24156.50	$z^{8}P_{4\frac{1}{2}}-9^{\circ}_{4\frac{1}{2}}$ $a^{8}D^{\circ}_{5\frac{1}{2}}-x^{8}F_{5\frac{1}{2}}$
4137.07	0	50	II	24164.90	$a^{8}D^{\circ}_{5\frac{1}{2}}-x^{8}F_{6\frac{1}{2}}$
4136.19	−1 −2	6	IV	24170.05	$a^{10}D^{\circ}_{2\frac{1}{2}}-133_{2\frac{1}{2}}$ $(a^{8}D^{\circ}_{5\frac{1}{2}}-x^{8}F_{4\frac{1}{2}})$
4128.10	0	8	IV	24217.41	$a^{10}D^{\circ}_{3\frac{1}{2}}-135_{2\frac{1}{2}}$
4127.28	+2	60	II	24222.22	$a^{8}D^{\circ}_{5\frac{1}{2}}-160_{4\frac{1}{2},5\frac{1}{2}}$
4123.85	−1	2	IV	24242.37	$z^{8}P_{4\frac{1}{2}}-f^{10}P^{\circ}_{4\frac{1}{2}}$
4117.016	0	15	IV	24282.61	$a^{10}D^{\circ}_{4\frac{1}{2}}-138_{3\frac{1}{2}}$
4115.65	+1	2	IV	24290.67	$z^{6}P_{3\frac{1}{2}}-14^{\circ}_{3\frac{1}{2},4\frac{1}{2}}$
4106.88	+1	60	II	24342.54	$a^{10}D^{\circ}_{2\frac{1}{2}}-135_{2\frac{1}{2}}$
4106.77	0		IV	24343.19	$a^{10}D^{\circ}_{5\frac{1}{2}}-142_{5\frac{1}{2}}$
4106.345	0	6	IV	24345.71	$z^{8}P_{3\frac{1}{2}}-f^{10}P^{\circ}_{3\frac{1}{2}}$
4102.702	0	25	III	24367.33	$a^{10}D^{\circ}_{4\frac{1}{2}}-140_{3\frac{1}{2}}$
4100.34	0	4	IV	24381.37	$z^{8}P_{4\frac{1}{2}}-10^{\circ}_{3\frac{1}{2},4\frac{1}{2}}$
4095.93	−2	5	IV	24407.61	$z^{8}P_{2\frac{1}{2}}-f^{10}P^{\circ}_{3\frac{1}{2}}$
4087.86	−1	8	IV	24455.80	$a^{10}D^{\circ}_{3\frac{1}{2}}-138_{3\frac{1}{2}}$
4078.24	0	40	II	24513.48	$a^{10}D^{\circ}_{6\frac{1}{2}}-149_{5\frac{1}{2},6\frac{1}{2}}$
4073.76	0	8	III	24540.44	$a^{10}D^{\circ}_{3\frac{1}{2}}-140_{3\frac{1}{2}}$
4071.20	0 +8	60	III	24555.87	$a^{8}D^{\circ}_{4\frac{1}{2}}-x^{8}F_{5\frac{1}{2}}$ $(a^{8}D^{\circ}_{4\frac{1}{2}}-x^{8}F_{3\frac{1}{2}})$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
4068.96	0	30	III	24569.39	$a^{8}D^{\circ}_{4\frac{1}{2}}-x^{8}F_{4\frac{1}{2}}$
4067.41	−1	3	IV	24578.75	$a^{10}D^{\circ}_{2\frac{1}{2}}-137_{2\frac{1}{2},\,3\frac{1}{2}}$
4065.422	0	20	III	24590.77	$a^{10}D^{\circ}_{4\frac{1}{2}}-143_{3\frac{1}{2}}$
4060.28	−2	15	III	24621.91	$a^{8}D^{\circ}_{4\frac{1}{2}}-160_{4\frac{1}{2},\,5\frac{1}{2}}$
4060.00	+1	20	IV	24623.61	$z^{8}P_{3\frac{1}{2}}-5^{\circ}_{3\frac{1}{2}}$
4059.03	+2	10	IV	24629.50	$a^{10}D^{\circ}_{4\frac{1}{2}}-144_{3\frac{1}{2}}$
4055.26	−1	4	IV	24652.39	$a^{8}D^{\circ}_{3\frac{1}{2}}-159_{2\frac{1}{2}}$
4053.087	0	10	IV	24665.61	$a^{10}D^{\circ}_{2\frac{1}{2}}-140_{3\frac{1}{2}}$
4052.08	+1	8	IV	24671.74	$z^{8}P_{3\frac{1}{2}}-6^{\circ}_{2\frac{1}{2},\,3\frac{1}{2}}$
4049.83	0	4	IV	24685.45	$z^{8}P_{2\frac{1}{2}}-5^{\circ}_{3\frac{1}{2}}$
4047.74	0	6	III	24698.19	$z^{8}P_{3\frac{1}{2}}-7^{\circ}_{3\frac{1}{2}}$
4043.97	−1	20	II	24721.22	$z^{10}P_{4\frac{1}{2}}-f^{10}D^{\circ}_{3\frac{1}{2}}$
4041.94	−1	15§	IV	24733.63	$z^{8}P_{2\frac{1}{2}}-6^{\circ}_{2\frac{1}{2},\,3\frac{1}{2}}$
4040.48	0	50	II	24742.57	$z^{10}P_{4\frac{1}{2}}-f^{10}D^{\circ}_{4\frac{1}{2}}$
4038.37	0	10	IV	24755.50	$a^{10}D^{\circ}_{6\frac{1}{2}}-152_{5\frac{1}{2},\,6\frac{1}{2}}$
4037.66	+2	10	IV	24759.85	$z^{8}P_{2\frac{1}{2}}-7^{\circ}_{3\frac{1}{2}}$
4036.99	−1	1	IV A	24763.96	$a^{10}D^{\circ}_{3\frac{1}{2}}-143_{3\frac{1}{2}}$
4036.55	0	20	IV	24766.66	$z^{8}P_{4\frac{1}{2}}-12^{\circ}_{4\frac{1}{2},\,5\frac{1}{2}}$
4036.15	0	50	II	24769.11	$z^{10}P_{4\frac{1}{2}}-f^{10}D^{\circ}_{5\frac{1}{2}}$
4034.51	0	1	IV	24779.18	$a^{8}D^{\circ}_{3\frac{1}{2}}-y^{6}F_{4\frac{1}{2}}$
4033.71	0	10	IV	24784.10	$z^{8}P_{4\frac{1}{2}}-13^{\circ}_{4\frac{1}{2},\,5\frac{1}{2}}$
4030.66	+4 −1	5	IV	24802.85	$a^{10}D^{\circ}_{6\frac{1}{2}}-y^{10}F_{5\frac{1}{2}}$ $a^{10}D^{\circ}_{3\frac{1}{2}}-144_{3\frac{1}{2}}$
4030.21	0	10	IV	24805.62	$a^{10}D^{\circ}_{5\frac{1}{2}}-147_{4\frac{1}{2}}$
4029.99	0 +1	150	II	24806.97	$a^{10}D^{\circ}_{6\frac{1}{2}}-y^{10}F_{7\frac{1}{2}}$ $(a^{10}D^{\circ}_{6\frac{1}{2}}-y^{10}F_{6\frac{1}{2}})$
4028.62	0	40	III	24815.41	$a^{8}D^{\circ}_{3\frac{1}{2}}-x^{8}F_{3\frac{1}{2}}$
4028.52	0 +1	10	IV	24816.02	$z^{8}P_{3\frac{1}{2}}-9^{\circ}_{4\frac{1}{2}}$ $(a^{8}D^{\circ}_{3\frac{1}{2}}-x^{8}F_{2\frac{1}{2}})$
4026.51	0	50	III	24828.41	$a^{8}D^{\circ}_{3\frac{1}{2}}-x^{8}F_{4\frac{1}{2}}$
4025.95	+1	25	IV	24831.86	$z^{8}P_{4\frac{1}{2}}-f^{10}P^{\circ}_{5\frac{1}{2}}$
4025.445	0	8	IV	24834.98	$a^{10}D^{\circ}_{5\frac{1}{2}}-149_{5\frac{1}{2},\,6\frac{1}{2}}$
4022.91	−1	6	IV	24850.63	$a^{10}D^{\circ}_{4\frac{1}{2}}-146_{3\frac{1}{2}}$
4016.694	0	125	II	24889.09	$a^{10}D^{\circ}_{2\frac{1}{2}}-143_{3\frac{1}{2}}$
4014.65	+1	12	IV	24901.76	$z^{8}P_{3\frac{1}{2}}-f^{10}P^{\circ}_{4\frac{1}{2}}$
4014.382	+1	25	III	24903.42	$a^{10}D^{\circ}_{5\frac{1}{2}}-151_{4\frac{1}{2}}$
4010.427	−1	40	III	24927.98	$a^{10}D^{\circ}_{2\frac{1}{2}}-144_{3\frac{1}{2}}$
4009.15	−1	1	IV	24935.92	$a^{8}D^{\circ}_{1\frac{1}{2}}-159_{2\frac{1}{2}}$
4002.90	+1	8	IV	24974.85	$a^{8}D^{\circ}_{2\frac{1}{2}}-x^{8}F_{1\frac{1}{2}}$
4000.81	0	20	III	24987.90	$a^{8}D^{\circ}_{2\frac{1}{2}}-x^{8}F_{3\frac{1}{2}}$
4000.70	0	30	III	24988.59	$a^{8}D^{\circ}_{2\frac{1}{2}}-x^{8}F_{2\frac{1}{2}}$
3995.74	0	8	IV	25019.60	$z^{8}P_{4\frac{1}{2}}-14^{\circ}_{3\frac{1}{2},\,4\frac{1}{2}}$
3995.07	−1	4	IV	25023.80	$a^{10}D^{\circ}_{3\frac{1}{2}}-146_{3\frac{1}{2}}$
3992.362	+2 0	8	II	25040.77	$z^{8}P_{3\frac{1}{2}}-10^{\circ}_{3\frac{1}{2},\,4\frac{1}{2}}$ $a^{10}D^{\circ}_{4\frac{1}{2}}-147_{4\frac{1}{2}}$
3987.90	+2	30	III	25068.79	$a^{10}D^{\circ}_{4\frac{1}{2}}-148_{3\frac{1}{2}}$
3987.10	−1	6	IV	25073.82	$a^{10}D^{\circ}_{6\frac{1}{2}}-156_{5\frac{1}{2}}$
3986.60	0	40	II	25076.96	$a^{10}D^{\circ}_{5\frac{1}{2}}-152_{5\frac{1}{2},\,6\frac{1}{2}}$
3986.08	−1	8	IV	25080.24	$a^{8}D^{\circ}_{1\frac{1}{2}}-x^{8}F_{\frac{1}{2}}$
3985.18	+1	10	IV	25085.90	$a^{8}D^{\circ}_{1\frac{1}{2}}-x^{8}F_{1\frac{1}{2}}$
3983.00	0	6	IV	25099.63	$a^{8}D^{\circ}_{1\frac{1}{2}}-x^{8}F_{2\frac{1}{2}}$
3979.19	−7	4§	IV	25123.66	$a^{10}D^{\circ}_{5\frac{1}{2}}-y^{10}F_{4\frac{1}{2}}$?

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
3979.02	−2	10	IV	25124.73	$a^{10}D^{\circ}_{5\frac{1}{2}}-y^{10}F_{5\frac{1}{2}}$
3978.42	0	50	II	25128.52	$a^{10}D^{\circ}_{5\frac{1}{2}}-y^{10}F_{6\frac{1}{2}}$
3976.832	+1	15	III	25138.56	$a^{10}D^{\circ}_{4\frac{1}{2}}-151_{4\frac{1}{2}}$
3975.20	0	4	IV	25148.88	$a^{10}D^{\circ}_{2\frac{1}{2}}-146_{3\frac{1}{2}}$
3969.90	+1	10	III	25182.45	$z^{10}P_{5\frac{1}{2}}-8^{\circ}_{4\frac{1}{2},\,5\frac{1}{2}}$
3969.22	−1	15	II	25186.77	$z^{10}P_{5\frac{1}{2}}-9^{\circ}_{4\frac{1}{2}}$
3967.18	+1	25	II	25199.72	$z^{10}P_{3\frac{1}{2}}-f^{10}D^{\circ}_{2\frac{1}{2}}$
3966.46	−1	5	IV	25204.29	$a^{10}D^{\circ}_{5\frac{1}{2}}-153_{4\frac{1}{2}}$
3964.95	0	60§	III	25213.89	$a^{10}D^{\circ}_{3\frac{1}{2}}-147_{4\frac{1}{2}}$
3964.49	+1	25	II	25216.82	$z^{10}P_{3\frac{1}{2}}-f^{10}D^{\circ}_{3\frac{1}{2}}$
3963.61	0	15	II	25222.41	$z^{8}P_{3\frac{1}{2}}-f^{6}D^{\circ}_{4\frac{1}{2}}$
3961.94	0	2	IV	25233.05	$z^{8}P_{3\frac{1}{2}}-f^{6}D^{\circ}_{3\frac{1}{2}}$
3961.12	0	20	II	25238.27	$z^{10}P_{3\frac{1}{2}}-f^{10}D^{\circ}_{4\frac{1}{2}}$
3960.52	−1	2	V	25242.09	$a^{10}D^{\circ}_{3\frac{1}{2}}-148_{3\frac{1}{2}}$
3955.75	0	80	II	25272.53	$z^{10}P_{5\frac{1}{2}}-f^{10}P^{\circ}_{4\frac{1}{2}}$
3952.25	−1	4	IV	25294.91	$z^{8}P_{2\frac{1}{2}}-f^{6}D^{\circ}_{3\frac{1}{2}}$
3949.84	+5	15	{IV V E}	25310.34	$z^{8}P_{2\frac{1}{2}}-f^{6}D^{\circ}_{2\frac{1}{2}}$
3949.60	−2	50	II	25311.88	$a^{10}D^{\circ}_{3\frac{1}{2}}-151_{4\frac{1}{2}}$
3944.05	−1	?	III	25347.50	$a^{10}D^{\circ}_{4\frac{1}{2}}-y^{10}F_{3\frac{1}{2}}$
3942.35	0	15	III	25358.43	$a^{10}D^{\circ}_{4\frac{1}{2}}-y^{10}F_{4\frac{1}{2}}$
3942.21	+7	30	{V E IV ?}	25359.33	$(a^{10}D^{\circ}_{4\frac{1}{2}}-y^{10}F_{5\frac{1}{2}})$
3936.643	+1	10	III	25395.19	$a^{10}D^{\circ}_{5\frac{1}{2}}-156_{5\frac{1}{2}}$
3929.81	0	15	III	25439.35	$a^{10}D^{\circ}_{4\frac{1}{2}}-153_{4\frac{1}{2}}$
3927.45	+1	10	III	25454.63	$a^{10}D^{\circ}_{4\frac{1}{2}}-154_{3\frac{1}{2}}$
3918.520	+1	40	II	25512.64	$a^{10}D^{\circ}_{6\frac{1}{2}}-158_{5\frac{1}{2},\,6\frac{1}{2}}$
3917.30	0	60	{V E II}	25520.59	$a^{10}D^{\circ}_{3\frac{1}{2}}-y^{10}F_{3\frac{1}{2}}$
3916.82	0	20	II	25523.71	$a^{10}D^{\circ}_{3\frac{1}{2}}-y^{10}F_{2\frac{1}{2}}$
3915.62	0	25	II	25531.53	$a^{10}D^{\circ}_{3\frac{1}{2}}-y^{10}F_{4\frac{1}{2}}$
3903.233	−1	20	II	25612.56	$a^{10}D^{\circ}_{3\frac{1}{2}}-153_{4\frac{1}{2}}$
3900.916	+1	10	II	25627.77	$a^{10}D^{\circ}_{3\frac{1}{2}}-154_{3\frac{1}{2}}$
3900.51	0	40	II	25630.44	$a^{10}D^{\circ}_{4\frac{1}{2}}-156_{5\frac{1}{2}}$
3898.75	0	30	II	25642.01	$a^{10}D^{\circ}_{2\frac{1}{2}}-y^{10}F_{1\frac{1}{2}}$
3898.18	0	10§	II	25645.76	$a^{10}D^{\circ}_{2\frac{1}{2}}-y^{10}F_{3\frac{1}{2}}$
3897.70	0	30	II	25648.92	$a^{10}D^{\circ}_{2\frac{1}{2}}-y^{10}F_{2\frac{1}{2}}$
3893.12	0	6	IV	25679.09	$z^{8}P_{3\frac{1}{2}}-14^{\circ}_{3\frac{1}{2},\,4\frac{1}{2}}$
3891.49	−1	8	IV	25689.84	$a^{10}D^{\circ}_{3\frac{1}{2}}-155_{3\frac{1}{2}}$
3884.75	0	100	III	25734.42	$z^{10}P_{4\frac{1}{2}}-f^{10}P^{\circ}_{3\frac{1}{2}}$
3881.92	−2	5	V ?	25753.18	$a^{10}D^{\circ}_{2\frac{1}{2}}-154_{3\frac{1}{2}}$
3875.34	0	30	III	25796.90	$z^{10}P_{5\frac{1}{2}}-12^{\circ}_{4\frac{1}{2},\,5\frac{1}{2}}$
3872.72	0	30	III	25814.35	$z^{10}P_{5\frac{1}{2}}-13^{\circ}_{4\frac{1}{2},\,5\frac{1}{2}}$
3869.75	0	80	III	25834.17	$a^{10}D^{\circ}_{5\frac{1}{2}}-158_{5\frac{1}{2},\,6\frac{1}{2}}$
3865.57	0	150	III	25862.10	$z^{10}P_{5\frac{1}{2}}-f^{10}P^{\circ}_{5\frac{1}{2}}$
3828.93	−4	30	{III V E}	26109.58	$z^{8}P_{4\frac{1}{2}}-15^{\circ}_{4\frac{1}{2},\,5\frac{1}{2}}$
3811.33	0	20	II	26230.14	$z^{10}P_{3\frac{1}{2}}-f^{10}P^{\circ}_{3\frac{1}{2}}$
3776.22	0	15	II	26474.01	$z^{10}P_{4\frac{1}{2}}-11^{\circ}_{3\frac{1}{2},\,4\frac{1}{2}}$
3774.10	0	150	III	26488.88	$z^{8}P_{4\frac{1}{2}}-16^{\circ}_{3\frac{1}{2},\,4\frac{1}{2}}$
3760.78	−1	10	II	26582.70	$z^{10}P_{3\frac{1}{2}}-7^{\circ}_{3\frac{1}{2}}$
3758.29	−1	30	V ?	26600.31	$z^{8}P_{4\frac{1}{2}}-17^{\circ}_{4\frac{1}{2},\,5\frac{1}{2}}?$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
3744.20	+1	30	II	26700.41	$z^{10}P_{3½}-9^{\circ}_{4½}$
3732.20	0	50	II	26786.26	$z^{10}P_{3½}-f^{10}P^{\circ}_{4½}$
3728.21	0	8	II	26814.93	$z^{10}P_{4½}-12^{\circ}_{4½,\,5½}$
3725.785	0	5	II	26832.38	$z^{10}P_{4½}-13^{\circ}_{4½,\,5½}$
3722.629	−1	100	IV	26855.13	$z^{8}P_{4½}-18^{\circ}$
3719.16	−1	100	II	26880.17	$z^{10}P_{4½}-f^{10}P^{\circ}_{5½}$
3706.8	−1	3	III ?	26969.8	$z^{10}P_{3½}-11^{\circ}_{3½,\,4½}$
3699.28	+1	2	III	27024.62	$a^{10}D^{\circ}_{3½}-159_{2½}$
3683.62	0	[4]	III	27139.51	$z^{10}P_{5½}-15^{\circ}_{4½,\,5½}$
3682.417	0	30	III	27148.38	$z^{8}P_{3½}-16^{\circ}_{3½,\,4½}$
3682.23	+1	3	III	27149.75	$a^{10}D^{\circ}_{2½}-159_{2½}$
3665.15	−2	6	III	27276.27	$a^{10}D^{\circ}_{6½}-162_{5½,\,6½}$
3652.14†	0	8§	IV	27373.44	$a^{10}D^{\circ}_{6½}-x^{10}F_{7½}$
3651.09	+2	4	III	27381.31	$a^{10}D^{\circ}_{5½}-161_{4½,\,5½}$
3648.86	0	5	III	27398.04	$a^{10}D^{\circ}_{6½}-x^{10}F_{5½}$
3647.88	+1	2	IV	27405.40	$a^{10}D^{\circ}_{6½}-x^{10}F_{6½}$
3627.99	+1	5	{III V E}	27555.64	$z^{8}P_{3½}-19^{\circ}_{3½,\,4½}?$
3622.48	+1	10?	III	27597.56	$a^{10}D^{\circ}_{5½}-162_{5½,\,6½}$
3619.96	−2	5	III	27616.77	$a^{10}D^{\circ}_{4½}-161_{4½,\,5½}$
3618.17	+1	25	II	27630.43	$z^{10}P_{5½}-17^{\circ}_{4½,\,5½}?$
3607.28	−1	8	III	27713.84	$a^{10}D^{\circ}_{5½}-x^{10}F_{4½}$
3606.54	0	10	II	27719.53	$a^{10}D^{\circ}_{5½}-x^{10}F_{5½}$
3605.57	−1	5	II	27726.99	$a^{10}D^{\circ}_{5½}-x^{10}F_{6½}$
3589.270	0	60	I	27852.90	$a^{8}S^{\circ}_{3½}-100_{3½,\,4½}$
3577.15	−1	8	III	27947.27	$a^{10}D^{\circ}_{4½}-x\ {}^{0}F_{3½}$
3576.94	+1	10	II	27948.91	$a^{10}D^{\circ}_{4½}-x^{10}F_{4½}$
3576.20	0	15	II	27954.69	$a^{10}D^{\circ}_{4½}-x^{10}F_{5½}$
3555.39	−1	5§	III	28118.31	$a^{10}D^{\circ}_{3½}-x^{10}F_{2½}$
3555.15	+1	6	III	28120.20	$a^{10}D^{\circ}_{3½}-x^{10}F_{3½}$
3554.91	0	8	III	28122.10	$a^{10}D^{\circ}_{3½}-x^{10}F_{4½}$
3539.78	0	4	III	28242.30	$a^{10}D^{\circ}_{2½}-x^{10}F_{1½}$
3539.65	+1	2	III	28243.34	$a^{10}D^{\circ}_{2½}-x^{10}F_{2½}$
3503.22	+1	20	IV	28537.03	$z^{10}P_{4½}-16^{\circ}_{3½,\,4½}$
3487.28	−2	2	II	28667.47	$a^{8}S^{\circ}_{3½}-z^{10}F_{2½}$
3467.880	0	30	I	28827.84	$a^{8}S^{\circ}_{3½}-101_{3½}$
3457.050	0	40§	I	28918.14	$a^{8}S^{\circ}_{3½}-z^{10}F_{3½}$
3453.89	−1	4	III	28944.60	$z^{10}P_{4½}-19^{\circ}_{3½,\,4½}?$
3432.520	0	20	I	29124.80	$a^{8}S^{\circ}_{3½}-103_{2½,\,3½}$
3353.705	0	20	I	29809.23	$a^{8}S^{\circ}_{3½}-104_{3½,\,4½}$
3350.403	0	50	I	29838.61	$a^{8}S^{\circ}_{3½}-105_{4½}$
3334.33	0	600r	I	29982.44	$a^{8}S^{\circ}_{3½}-106_{3½,\,4½}$
3322.263	0	80	I	30091.34	$a^{8}S^{\circ}_{3½}-107$
3262.50	0	10	I	30642.54	$a^{8}S^{\circ}_{3½}-108_{3½,\,4½}$
3247.550	+1	40	I	30783.59	$a^{8}S^{\circ}_{3½}-109_{3½}$
3246.032	0	20	I	30797.99	$a^{8}S^{\circ}_{3½}-z^{8}D_{2½}$
3241.405	−1	50	I	30841.95	$a^{8}S^{\circ}_{3½}-z^{8}D_{3½}$
3235.126	−1	30	I	30901.81	$a^{8}S^{\circ}_{3½}-z^{8}D_{4½}$
3230.61	+1	5	II	30945.00	$a^{8}S^{\circ}_{3½}-z^{10}D_{2½}$
3213.75	−1	200	II	31107.34	$a^{8}S^{\circ}_{3½}-112_{2½}$
3212.81	−1	1000R	II	31116.44	$a^{8}S^{\circ}_{3½}-113_{3½}$
3210.57	0	300	II	31138.15	$a^{8}S^{\circ}_{3½}-z^{10}D_{3½}$

TABLE 7—*Continued*

λ	Δλ*	Arc. Int.	Temp. Class	Wave No. Vac.	Multiplet Designation
3185.54	−3	70	II V E	31382.81	$a^8S^\circ_{3\frac{1}{2}}-z^{10}D_{4\frac{1}{2}}$
3168.282	0	30	I	31553.75	$a^8S^\circ_{3\frac{1}{2}}-114_{4\frac{1}{2}}$
3111.43	0	500R	I	32130.27	$a^8S^\circ_{3\frac{1}{2}}-115_{4\frac{1}{2}}$
3106.18	0	100	I	32184.58	$a^8S^\circ_{3\frac{1}{2}}-z^8F_{4\frac{1}{2}}$
3085.70	0	6	II	32398.18	$a^8S^\circ_{3\frac{1}{2}}-y^{10}P_{3\frac{1}{2}}$
3066.950	0	40	I	32596.24	$a^8S^\circ_{3\frac{1}{2}}-y^{10}P_{4\frac{1}{2}}$
3058.984	0	100	I	32681.12	$a^8S^\circ_{3\frac{1}{2}}-118_{3\frac{1}{2}}$
2958.91	+1	30	I	33786.39	$a^8S^\circ_{3\frac{1}{2}}-x^8P_{2\frac{1}{2}}$
2950.82	+1	20	I	33879.01	$a^8S^\circ_{3\frac{1}{2}}-119_{3\frac{1}{2}}$
2948.23	0	4	II V	33908.77	$a^8S^\circ_{3\frac{1}{2}}-120_{3\frac{1}{2}}$
2931.52	−2	6	V E ?	34102.05	$a^8S^\circ_{3\frac{1}{2}}-x^8P_{3\frac{1}{2}}$
2908.99	0	250	I	34366.15	$a^8S^\circ_{3\frac{1}{2}}-w^8P_{4\frac{1}{2}}$
2893.83	−1	300	I	34546.18	$a^8S^\circ_{3\frac{1}{2}}-x^{10}P_{4\frac{1}{2}}$
2893.03	+1	150	I	34555.73	$a^8S^\circ_{3\frac{1}{2}}-w^8P_{2\frac{1}{2}}$
2892.54	+1	200	I	34561.58	$a^8S^\circ_{3\frac{1}{2}}-w^8P_{3\frac{1}{2}}$
2878.87	0	50	I	34725.69	$a^8S^\circ_{3\frac{1}{2}}-x^8P_{4\frac{1}{2}}$
2877.784	0	15	I	34738.79	$a^8S^\circ_{3\frac{1}{2}}-122_{4\frac{1}{2}}$
2807.20	+1	4	I	35612.22	$a^8S^\circ_{3\frac{1}{2}}-123_{3\frac{1}{2}}$
2800.04	+2	5	I	35703.28	$a^8S^\circ_{3\frac{1}{2}}-124_{2\frac{1}{2}}$
2797.81	−1	2	II	35731.73	$a^8S^\circ_{3\frac{1}{2}}-z^6F_{4\frac{1}{2}}$
2776.52	0	10	I	36005.70	$a^8S^\circ_{3\frac{1}{2}}-x^{10}P_{3\frac{1}{2}}$
2772.90	−1	10	II	36052.70	$a^8S^\circ_{3\frac{1}{2}}-128_{2\frac{1}{2}}$
2771.44	0	3	II	36071.70	$a^8S^\circ_{3\frac{1}{2}}-129_{3\frac{1}{2},\,4\frac{1}{2}}$
2770.73	0	2	II	36080.94	$a^8S^\circ_{3\frac{1}{2}}-z^6F_{3\frac{1}{2}}$
2747.83	−1	60	I	36381.61	$a^8S^\circ_{3\frac{1}{2}}-v^8P_{3\frac{1}{2}}$
2745.61	−1	40	II	36411.03	$a^8S^\circ_{3\frac{1}{2}}-v^8P_{2\frac{1}{2}}$
2743.28	−1	80	II	36441.95	$a^8S^\circ_{3\frac{1}{2}}-u^8P_{2\frac{1}{2}}$
2738.57	−1	15	II	36504.63	$a^8S^\circ_{3\frac{1}{2}}-u^8P_{4\frac{1}{2}}$
2735.254	0	40	II	36548.88	$a^8S^\circ_{3\frac{1}{2}}-v^8P_{4\frac{1}{2}}$
2732.61	0	18	II	36584.24	$a^8S^\circ_{3\frac{1}{2}}-y^8D_{2\frac{1}{2}}$
2731.37	0	20	I	36600.85	$a^8S^\circ_{3\frac{1}{2}}-u^8P_{3\frac{1}{2}}$
2723.96	−1	50	II	36700.41	$a^8S^\circ_{3\frac{1}{2}}-y^8D_{3\frac{1}{2}}$
2709.99	0	40	II	36889.59	$a^8S^\circ_{3\frac{1}{2}}-y^8D_{4\frac{1}{2}}$
2695.08	+1	2	II	37093.66	$a^8S^\circ_{3\frac{1}{2}}-133_{2\frac{1}{2}}$
2692.74	+1	5	II	37125.89	$a^8S^\circ_{3\frac{1}{2}}-134_{4\frac{1}{2}}$
2682.60	+1	8	II	37266.22	$a^8S^\circ_{3\frac{1}{2}}-135_{2\frac{1}{2}}$
2659.42	+1	15	III	37591.01	$a^8S^\circ_{3\frac{1}{2}}-141_{4\frac{1}{2}}$
2643.84	+2	6	V	37812.52	$a^8S^\circ_{3\frac{1}{2}}-143_{3\frac{1}{2}}$
2637.14	+1	3	V	37908.58	$a^8S^\circ_{3\frac{1}{2}}-145_{4\frac{1}{2}}$
2625.79	+1	2	V	38072.43	$a^8S^\circ_{3\frac{1}{2}}-146_{3\frac{1}{2}}$
2619.27	−1	10	V	38167.20	$a^8S^\circ_{3\frac{1}{2}}-z^6D_{4\frac{1}{2}}$
2609.8	−3	1	V	38305.7	$a^8S^\circ_{3\frac{1}{2}}-150_{3\frac{1}{2}}$
2606.1	+4	1	V	38360.1	$a^8S^\circ_{3\frac{1}{2}}-151_{4\frac{1}{2}}$
2602.6	+1	3	V	38411.6	$a^8S^\circ_{3\frac{1}{2}}-z^6D_{3\frac{1}{2}}$
2585.76	−3	12	V E ?	38661.79	$a^8S^\circ_{3\frac{1}{2}}-153_{4\frac{1}{2}}$
2580.62	−1	3	V	38738.79	$a^8S^\circ_{3\frac{1}{2}}-155_{3\frac{1}{2}}$
2564.98	0	4	V	38974.98	$a^8S^\circ_{3\frac{1}{2}}-157_{4\frac{1}{2}}$
2428.2†	+4 −7	1	V	41170.3	$a^8S^\circ_{3\frac{1}{2}}-x^{10}F_{4\frac{1}{2}}$? $a^8S^\circ_{3\frac{1}{2}}-x^{10}F_{3\frac{1}{2}}$?

The average difference, regardless of sign, between the observed and computed wave lengths, is 0.0085 A for 248 lines of intensity greater than 50 and upward; 0.0095 for 237 lines of intensity 11–50; and 0.0151 for 459 lines of intensity 10 or less. When n lines have been used to fix the value of one of the upper levels, the average residual should be $1.17\{n/(n-1)\}^{\frac{1}{2}}$ times the probable error of a

TABLE 8

STRONGEST UNCLASSIFIED LINES OF *Eu* I

λ	Intensity Arc	Intensity Furnace	Temp. Class	λ	Intensity Arc	Intensity Furnace	Temp. Class
3629.8	30	......	V	4575.79	20	......	V
3641.27	20	......	V	4586.38	30	20	III
3646.75	35	......	V	4641.41	20	......	V
3656.19	20	10	II	4644.23	50	......	V
3696.42	20	......	V	4651.545	20	......	V
3735.94	20	......	V E?	4653.30	20	15	IV
3746.05	20	1	IV?	4665.07	20	......	V
3916.00	50	30	II	4671.166	25	......	V
4039.19	200	100	II	4758.74	40	30	III
4103.87	20	8	III	5864.77	40	40	III
4125.53	20	8	IV	6360.48	3	20	III A
4249.40	20	15	IV				
4325.53	30	20	III				
4405.27	20	4	IV				
4406.79	20	1	IV				

measure (provided that the low levels were exactly known). The values of n range from 12 to 1. For the 229 levels here considered, the mean value of n is 4.73, while that of $\{n/(n-1)\}^{\frac{1}{2}}$ is 0.794. For the 33 low levels, the mean value of n is 32.8 and that of $\{n/(n-1)\}^{\frac{1}{2}}$ is 0.982. Hence (closely enough) the average residual should be 0.91 times the probable error, which is thus found to be ± 0.0094 A for lines of intensity exceeding 50, ± 0.0105 for intensities between 10 and 50, and ± 0.0166 for the fainter lines. As the latter include many which were barely visible and difficult to measure, the increase is not surprising.

The lines remaining unclassified to the red of λ 3500,[6] for which the estimated intensity either in the arc or in the furnace is 20 or more, are listed in Table 8. There are 26 of them, as against 494 classified lines in the same spectral region above the same limit of intensity. The one outstanding line, λ 4039, is probably the strongest line of some multiplet—a transition from the leading component of a term like $a^{10}D^o$ to a higher level of large *J*-value, from which no other combinations are to be expected. The very strong line λ 5830.98, $a^{10}D^o_{6\frac{1}{2}} - z^{10}F_{7\frac{1}{2}}$, is of this sort.

The remaining low-temperature lines probably represent transitions from $a^{10}D^o$ and a^8D^o to additional levels similar to those which have received numbers in Table 6, while the lines of class V are discussed at the end of section 3.

Complete identification of the lines of so complex a spectrum could hardly be anticipated, especially in view of the rarity of the material and the absence of Zeeman data. Enough has been done, however, to show that the spectrum of a rare earth is fully amenable to the existing methods of analysis. The spectra of even the neighboring elements, samarium and gadolinium, should theoretically be far more intricate;[1] but, with sufficiently precise wave-lengths, Zeeman data, and, above all, with labor enough in searching, they should yield as fully to analysis.

In the previous publication[4] note is made of the identification in the sunspot spectrum of two of the ultimate lines of *Eu* I (the third being masked), and in the solar spectrum of some additional lines of *Eu* II.

In conclusion, it is a pleasure to express our gratitude to Miss C. E. Moore for extensive aid in the tabulations and proofreading and for a critical examination of the whole manuscript. The accuracy of the results is mainly due to her careful work.

PRINCETON UNIVERSITY OBSERVATORY,
CARNEGIE INSTITUTION OF WASHINGTON
MOUNT WILSON OBSERVATORY
March 1939

[6] For shorter wave lengths the separation of arc and spark lines is often difficult.

NOTES

ON THE ORIGIN OF EMISSION IN γ CASSIOPEIAE

Possible mechanisms for the production of emission lines in the extended atmospheres of B-type stars are collisions, excitation, or ionization and subsequent recombination. It is probable that the effects of collisions may be neglected, as the density in the outer atmosphere must be extremely low. Excitation is an inefficient method for the production of emission. On the other hand, if the atom is first ionized and then undergoes recombination with a free electron, the emission lines may be rather strong.

In γ Cassiopeiae both of the latter mechanisms seem to be active. During the single-line stage approximately two hundred emission structures were visible in the spectrum from $\lambda 4924$ to the head of the Balmer series.[1] About fifty of these lines have as yet received no definite identifications. However, Swings and Edlén[2] have recently identified three of the lines as *Fe* III. The ionization potential of neutral iron is 7.83 volts, of singly ionized iron 16.5 volts, and of doubly ionized iron about 30 volts. Consequently to triply ionize iron atoms we must apply something of the order of 50 volts. This is incompatible with the atmospheric energy conditions shown by the rest of the spectrum, and therefore we may assume that excitation is the sole method by which the emission lines of *Fe* III are formed. The presence of the *Fe* III lines is important as an indication that a very appreciable fraction of the iron atoms are doubly ionized and that the strong *Fe* II spectrum is largely one of recombination.

An even more striking confirmation of recombination as the source of the emission lines may be found in the Balmer hydrogen spectrum. If the lower Balmer lines are produced mainly by excitation, the lines near the limit of the series will be extremely weak if, indeed, they are present at all in emission. By contrast, if the lines are produced by recombination, even the higher members of the

[1] Baldwin, *Ap. J.*, **87**, 573, 1938. [2] *Ap. J.*, **88**, 618, 1938.

Balmer series will appear, and there will be a definite emission continuum at the series limit and extending toward the shorter wave lengths.

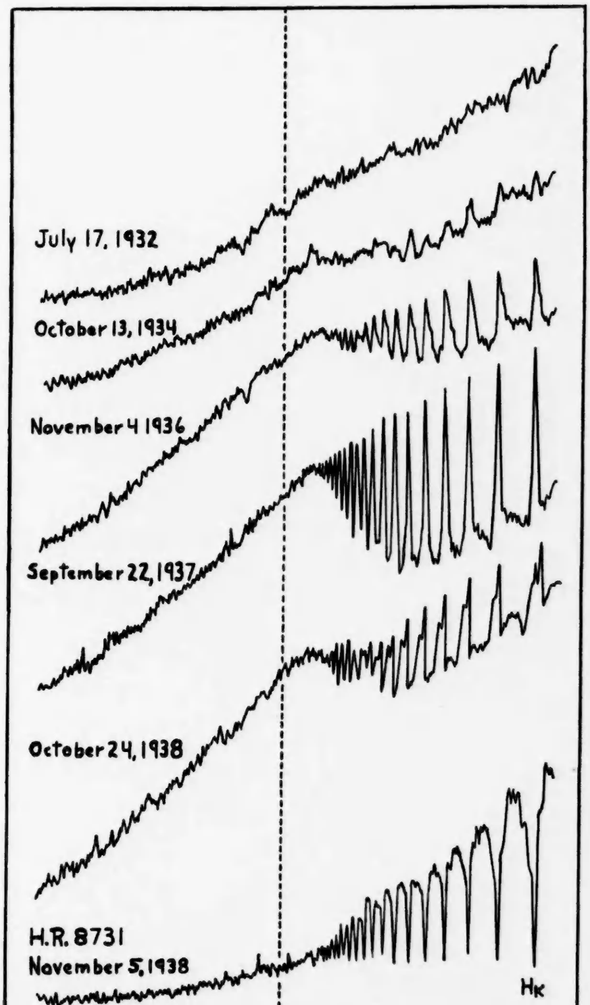

Fig. 1.—Microphotometer tracings of the spectrum of γ Cassiopeiae

In Figure 1 the Balmer lines of shorter wave lengths than $H\kappa$ are shown for several dates. On the plate of 1932, emission lines are found out as far as $H\iota$, while $H\kappa$ and all higher members are

neutral, indicating an approximate equality of emission and absorption. That emission was present is shown by the suggestion of an emission continuum.

On later dates the Balmer lines became more and more prominent, reaching a maximum during the single-line stage in 1937.[1] The emission continuum also became tremendously strong, thus paralleling the rise of the line intensities. After the single-line stage the emission intensity, both of the lines and of the continuum, declined immediately, although both were still many times as strong in 1938 as in 1932. Because the Balmer lines are not monochromatic in γ Cassiopeiae, the higher members of the series become blended before the series limit (dotted line) is reached.

Unfortunately, these spectra are not suited to quantitative analysis of the continuum as the atmospheric, instrumental, and plate corrections are not known. They do, however, show a definite rise and fall of emission strength of both the lines and the continuum. The presence of the latter on each plate can be accounted for only by the process of ionization and recombination.

All spectrograms were made on Process emulsion with the $37\frac{1}{2}$-inch reflector of the University of Michigan Observatory. The first two were made with a silver-on-glass mirror and the last four with an aluminum-on-glass mirror. A microphotometer tracing of the spectrum of the absorption-line B star, HR 8731, is shown for comparison. There is no indication of a hump in the plate sensitivity-curve near the limit of the Balmer series. The tracings were made with a low-resolution microphotometer at the physics department of Northwestern University.

RALPH B. BALDWIN

DEARBORN OBSERVATORY
NORTHWESTERN UNIVERSITY
June 10, 1939

REVIEWS

Astronomy. By W. T. Skilling and R. S. Richardson. New York: Henry Holt & Co., 1939. Pp. 579. $3.00.

Despite its all-embracing title, this book fills a specific present-day demand for an interesting description of astronomy, not too difficult for a student with very little scientific background, yet fairly complete and up to date in its treatment of the latest advances in the subject. The authors have had access to some of the best illustrations ever published, to which they have added many helpful diagrams.

In general, the text presents the problems of astronomy without their mathematical complexity. Thus, for instance, the rotation of the galaxy, the velocity of light, the diameters of stars determined by the interferometer, and the aberration of light are all presented in terms that can be clearly understood by the neophyte. On the other hand, this treatment permits of inaccuracies and a certain amount of vagueness. Professional astronomers will be annoyed by this and by perhaps too many appeals to the reader's intuition; philosophers will probably criticize the pragmatic tone; but large numbers of elementary students will no doubt have their interest captured by the simple and informal style.

The order of the material is definitely peculiar, and the smoothness of the text pays tribute to the authors' writing ability. The knowing reader may be surprised in jumping from the earth to the celestial sphere, back to instruments, on to the physics of the sun, and back to the confusing question of time. One peculiar insertion in chapter xv is a short description of constellations, lost between discussions of distance and proper motions. Owing to an omission, the reader puzzles over "magnitude" and "spectral type" for several chapters before their definition in chapter xvi.

However, teachers and students alike will be pleased with the sketches of early telescopes, with the interesting accounts of casting the 200-inch disk, of the Schmidt camera, of the thermoelectric and photoelectric cells, of eclipse expeditions, of the motions of eruptive prominences, of eclipse limits, of theories of the origin of the solar systems, of direct and opposite tides, of supergiant and white dwarf stars, of the velocity-distance relation for spirals, and with the extensive glossary of terms before the Index.

Problems at the end of each chapter will help greatly to solidify the careful student's understanding of the material.

From their fairly exhaustive treatment the authors have unfortunately omitted a clear explanation of relativity which, as usual, is mentioned as the mysterious Einstein theory "requiring" this or that. With some exposition of just the philosophy of special relativity a much more complete section on cosmogony could have been incorporated. Other, less serious, omissions are: (1) the theory of nebular luminosity, (2) the ozone and ionized layers of the atmosphere, (3) polarized light, (4) star catalogues, and (5) the importance of dynamical parallaxes.

In spite of the inevitable criticism levied at any elementary exposition of recent research, it may be said that this is the best single book available for teaching a large amount of astronomy to the uninitiated student.

THORNTON PAGE

ERRATUM

The writer regrets that in his paper on "The Spectra of Bright Chromospheric Eruptions from λ 3300 to λ 11500" (*Ap. J.*, **89**, 1939) he stated on page 348 that the records of the bright eruptions are sent to Zurich, which acts as a clearing house for these data.

On the contrary, they are sent to Dr. L. d'Azumbuja at Meudon, who for years has performed the valuable service of organizing this mass of material into a form suitable for publication.

R. S. RICHARDSON

CARNEGIE INSTITUTION OF WASHINGTON
MOUNT WILSON OBSERVATORY
June 1939